Assignment 1, MATH 6102

1. For the spruce budworm population model

$$\frac{du}{dt} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}, \tag{1}$$

where r and q are positive dimensionless parameters, show that the curve in r, q space which divides it into regions where there are one or three positive steady states is given parametrically by $r = \frac{2a^3}{(1+a^2)^2}$, $q = \frac{2a^3}{a^2-1}$, and then sketch the curves in r, q space noting the limiting behavior of $r(a)$ and $q(a)$ as $a \to \infty$ and $a \to 1$.

2. Consider the periodic logistic equation

$$\frac{du}{dt} = u\left(b(t) - a(t)u\right), \tag{2}$$

where $a(t)$ and $b(t)$ are ω-periodic functions. Assume that $a(t) > 0$, $\forall t \geq 0$, and $\int_0^\omega b(t)dt > 0$. Find an explicit expression of the solution $u(t)$ to equation (2) with $u(0) = u_0 > 0$, and then show that $u(t)$ is asymptotic to a unique positive ω-periodic solution $u^*(t)$.

3. Consider the periodic ODE system

$$\frac{du}{dt} = F(t, u), \tag{3}$$

where $F \in C^1([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ is ω-periodic in t for some $\omega > 0$. Assume that $u(t)$ is a solution of system (3) on $[0, \infty)$ such that $\lim_{n\to\infty} u(n\omega) = u^*$. Show that the unique solution $u^*(t)$ of system (3) with $u^*(0) = u^*$ is ω-periodic in t, and $\lim_{t\to\infty}(u(t) - u^*(t)) = 0$.