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GLOBAL ATTRACTORS AND STEADY STATES FOR UNIFORMLY PERSISTENT DYNAMICAL SYSTEMS*

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Abstract. By appealing to the theory of global attractors on complete metric spaces, we obtain weaker sufficient conditions for the existence of interior global attractors for uniformly persistent dynamical systems, and hence generalize the earlier results on coexistence steady states. We also provide examples to show applicability of our interior fixed point theorem in the case of convex κ-contracting maps, and to prove the existence of discrete- and continuous-time dynamical systems that admit global attractors, but no strong global attractors, which gives an affirmative answer to an open question presented by Sell and You [*Dynamics of Evolutionary Equations*, Springer-Verlag, New York, 2002] in the case of continuous-time semiflows.

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1. Introduction.

Uniform persistence is an important concept in population dynamics since it characterizes the long-term survival of some or all interacting species in an ecosystem. There have been extensive investigations on uniform persistence for discrete- and continuous-time dynamical systems. We refer to [13, 27, 30] for surveys and reviews. Looked at abstractly, uniform persistence is the notion that a closed subset of the state space (e.g., the set of extinction for one or more populations) is repelling for the dynamics on the complementary set. A natural question concerns the existence of "interior" global attractors and "coexistence" steady states for uniformly persistent dynamical systems. The existence of interior global attractors was addressed by Hale and Waltman [10], and the existence of coexistence steady states under a general setting was investigated by Zhao [29]. In [10, 29] the traditional concept of global attractors was employed: a global attractor is a compact, invariant set which attracts every bounded set in the phase space (see, e.g., Hale [7], Temam [24], and Raugel [20]).

Recently, the following weaker concept of global attractors was introduced by Hirsch, Smith, and Zhao [11] and Sell and You [22]: a global attractor is a compact, invariant set which attracts some neighborhood of itself and every point in the phase space. For convenience, we refer to a traditional global attractor as a strong global attractor. With the concept of strong global attractor, Zhao [29, Theorem 2.3] assumed more conditions than necessary for the existence of a coexistence fixed point. However, the proof of [29, Theorem 2.3] needs only the property that the interior attractor attracts every compact set, and hence actually implies a general fixed point theorem that, if a continuous and κ-condensing map T has an interior global attractor, then it has a coexistence fixed point (see Theorem 4.1). So an important problem is to obtain sufficient conditions for the existence of interior global attractors for uniformly

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persistent dynamical systems. This is a nontrivial problem since the phase space M_0 is an open subset of a complete metric space (M, d).

The main purpose of this paper is to establish the existence of the interior global attractor (i.e., the global attractor for $T : (M_0, d) \to (M_0, d)$) and a fixed point in M_0.

There is the following open question on the weaker concept of global attractor (see pages 55 and 56 of [22]): Does there exist an example of a κ-contracting semiflow that is point dissipative on a complete metric space W in which the global attractor does not attract every bounded set in W? In other words, we expect to find a dynamical system which has a global attractor, but no strong global attractor. In the case of discrete-time semiflows (i.e., maps), such a question has already been answered positively by Cholewa and Hale [4] (see also Raugel [20]) who developed an original result of Cooperman [5] and introduced an appropriate κ-contraction map on the Hilbert space of square summable series. As a byproduct of our investigations on interior global attractors, we will provide examples of both discrete- and continuous-time semiflows to give an affirmative answer to Sell and You's question (see sections 5.2 and 5.3). It is worth pointing out that our continuous-time dynamical systems are solution semiflows associated with a class of evolutionary equations with age structure.

It is obvious that we should start with the development of the theory of strong global attractors into that of global attractors on complete metric spaces. Note that the metric space (M_0, d) is not complete since M_0 is an open subset of the complete metric space (M, d). In order to apply the theory of global attractors to $T : (M_0, d) \to (M_0, d)$, we introduce a new metric $d_0(x, y)$ on M_0 (see equation (2) for its definition) so that (M_0, d_0) is a complete metric space. It turns out that the strongly bounded sets introduced in [10] correspond to the bounded sets in (M_0, d_0). This metric function $d_0(x, y)$ is the key tool for both the existence of a global attractor for $T : (M_0, d) \to (M_0, d)$ and four counterexamples of dynamical systems on the complete metric space (M_0, d_0). The theory of global attractors was already developed for continuous-time semiflows in the book [22], where the concept of κ-contracting maps was introduced (i.e., for each bounded set $B \subset M$, $\kappa(T^n(B)) \to 0$ as $n \to +\infty$). It seems that this strong notion may not be applied to $T : (M_0, d_0) \to (M_0, d_0)$. In fact, if $T : (M_0, d_0) \to (M_0, d_0)$ is κ-contracting, then $T^n(B)$ is strongly bounded for all sufficiently large integers n whenever B is a strongly bounded subset of M_0. But this property may not be satisfied in general in the applications (see the first example in section 5.1). So we will use the concept of asymptotically smooth maps introduced in [7] to establish the existence of global attractors.

Using our established theory of global attractor in M_0, we further investigate the existence of a fixed point of T in M_0. We also generalize the aforementioned coexistence fixed point theorem for κ-condensing maps to convex κ-contracting maps (see Definition 4.3), a new concept motivated by the fixed point theorem of Hale and Lopes [9] and the Poincaré maps associated with periodic age-structured population models. In terms of uniform persistence, our fixed point theorem (see Theorem 4.5) and its corollary (see Corollary 4.6) generalize earlier results due to Browder [2], Nussbaum [18, 19], Zhao [29], and Magal and Arino [15]. Clearly, there are analogues of interior global attractors and fixed point results for continuous-time semiflows (see Remark 3.10 and Theorem 4.7).

This paper is organized as follows. In section 2, we recall some basic concepts and results for dissipative dynamical systems based on the book of Hale [7] and establish sufficient conditions for the existence of global attractors and strong global attractors. In section 3, we prove the existence of a global attractor for $T : (M_0, d) \to (M_0, d)$. In section 4, we present the fixed point theorems and their corollaries. In section 5, we provide four examples to show the existence of discrete- and continuous-time dynamical systems that admit global attractors, but no strong global attractors. A simple periodic age-structured model is also studied in this section to illustrate applicability of Theorem 4.5 in the case of convex κ-contracting maps.

2. Preliminaries. Let (M, d) be a complete metric space. Recall that a set U in M is said to be a neighborhood of another set V provided V is in the interior $\mathrm{int}(U)$ of U. For any subsets A, $B \subset M$ and any $\epsilon > 0$, we define

$$d(x, A) := \inf_{y \in A} d(x, y), \quad \delta(B, A) := \sup_{x \in B} d(x, A),$$

$$N(A, \epsilon) := \{x \in M : d(x, A) < \epsilon\} \text{ and } \overline{N}(A, \epsilon) := \{x \in M : d(x, A) \leq \epsilon\}.$$

The *Kuratowski measure of noncompactness*, κ, is defined by

$$\kappa(B) = \inf\{r : B \text{ has a finite open cover of diameter } \leq r\}$$

for any bounded set B of M. We set $\kappa(B) = +\infty$ whenever B is unbounded.

For various properties of Kuratowski's measure of noncompactness, we refer to [17, 6] and [22, Lemma 22.2]. The proof of the following lemma is straightforward.

LEMMA 2.1. *The following statements are valid:*

(a) *Let $I \subset [0, +\infty)$ be unbounded, and let $\{A_t\}_{t \in I}$ be a nonincreasing family of nonempty closed subsets (i.e., $t \leq s$ implies $A_s \subset A_t$). Assume that $\kappa(A_t) \to 0$, as $t \to +\infty$. Then $A_\infty = \cap_{t \geq 0} A_t$ is nonempty and compact, and $\delta(A_t, A_\infty) \to 0$, as $t \to +\infty$.*

(b) *For each $A \subset M$ and $B \subset M$, we have $\kappa(B) \leq \kappa(A) + \delta(B, A)$.*

Let $T : M \to M$ be a continuous map. We consider the discrete-time dynamical system $T^n : M \to M$ $\forall n \geq 0$, where $T^0 = Id$ and $T^n = T \circ T^{n-1}$ $\forall n \geq 1$. We denote for each subset $B \subset M$, $\gamma^+(B) = \cup_{m \geq 0} T^m(B)$ the *positive orbit of* B *for* T, and denote

$$\omega(B) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} T^m(B)}$$

the *omega-limit set of* B. A subset $A \subset M$ is *positively invariant* for T if $T(A) \subset A$. A is *invariant* for T if $T(A) = A$. We say that a subset $A \subset M$ *attracts* a subset $B \subset M$ for T if $\lim_{n \to \infty} \delta(T^n(B), A) = 0$.

It is easy to see that B is precompact (i.e., $\overline{B}$ is compact) if and only if $\kappa(B) = 0$. A continuous mapping $T : X \to X$ is said to be *compact* (*completely continuous*) if T maps any bounded set to a precompact set in M.

The theory of attractors is based on the following fundamental result, which is related to [7, Lemmas 2.1.1 and 2.1.2].

LEMMA 2.2. *Let B be a subset of M, and assume that there exists a compact subset $C \subset M$, which attracts B for T. Then $\omega(B)$ is nonempty, compact, invariant for T, and attracts B.*

Proof. Let $I = N$, the set of all nonnegative integers, and

$$A_n = \overline{\bigcup_{m \geq n} T^m(B)} \ \forall n \geq 0.$$

Since C attracts B, from Lemma 2.1(b) we deduce that

$$\kappa(A_n) \leq \kappa(C) + \delta(A_n, C) = \delta(A_n, C) \to 0 \text{ as } n \to +\infty.$$

So the family $\{A_n\}_{n\geq 0}$ satisfies the conditions of assertion (a) in Lemma 2.1, and we deduce that $\omega(B)$ is nonempty, compact, and $\delta(A_n, \omega(B)) \to 0$, as $n \to +\infty$. So $\omega(B)$ attracts B for T. Moreover, we have

$$T\left(\bigcup_{m\geq n} T^m(B)\right) = \bigcup_{m\geq n+1} T^m(B) \ \forall n \geq 0,$$

and since T is continuous, we obtain

$$T(A_n) \subset A_{n+1}, \text{ and } A_{n+1} \subset \overline{T(A_n)} \ \forall n \geq 0.$$

Finally, since $\delta(A_n, \omega(B)) \to 0$, as $n \to +\infty$, we have $T(\omega(B)) = \omega(B)$. □

Definition 2.3. *A continuous mapping $T : M \to M$ is said to be* point (compact, bounded) dissipative *if there is a bounded set B_0 in M such that B_0 attracts each point (compact set, bounded set) in M; T is* κ-condensing (κ-contraction *of order k, $0 \leq k < 1$) if T takes bounded sets to bounded sets and $\kappa(T(B)) < \kappa(B)(\kappa(T(B)) \leq k\kappa(B))$ for any nonempty closed bounded set $B \subset M$ with $0 < \kappa(B) < +\infty$; T is* asymptotically smooth *if, for any nonempty closed bounded set $B \subset M$ for which $T(B) \subset B$, there is a compact set $J \subset B$ such that J attracts B.*

Clearly, a compact map is a κ-contraction of order 0, and a κ-contraction of order k is κ-condensing. It is well known that κ-condensing maps are asymptotically smooth (see, e.g., [7, Lemma 2.3.5]). By Lemma 2.1, it follows that $T : M \to M$ is asymptotically smooth if and only if $\lim_{n\to\infty} \kappa(T^n(B)) = 0$ for any nonempty closed bounded subset $B \subset M$ for which $T(B) \subset B$.

A positively invariant subset $B \subset M$ for T is said to be *stable* if for any neighborhood V of B there exists a neighborhood $U \subset V$ of B such that $T^n(U) \subset V$ $\forall n \geq 0$. We say that A is *globally asymptotically stable* for T if, in addition, A attracts points of M for T.

By the proof that (i) implies (ii) in [7, Theorem 2.2.5], we have the following result.

Lemma 2.4. *Let $B \subset M$ be compact and positively invariant for T. If B attracts compact subsets of one of its neighborhoods, then B is stable.*

Definition 2.5. *A nonempty, compact, and invariant set $A \subset M$ is said to be an* attractor for T *if A attracts one of its neighborhoods; a* global attractor for T *if A is an attractor that attracts every point in M; and a* strong global attractor for T *if A attracts every bounded subset of M.*

We remark that the notions of attractor and global attractor were used in [11, 22, 30]. The strong global attractor was defined as a global attractor in [7, 24, 20]. The following result is essentially the same as [8, Theorem 3.2]. Note that the proof of this result was not provided in [8]. For completeness, we state it in terms of global attractors and give an elementary proof below.

Theorem 2.6. *Let T be a continuous map on a complete metric space (M, d). Assume that*

(a) *T is point dissipative and asymptotically smooth;*

(b) *positive orbits of compact subsets of M for T are bounded.*

Then T has a global attractor $A \subset M$. Moreover, for each subset B of M, if there exists $k \geq 0$ such that $\gamma^+(T^k(B))$ is bounded, then A attracts B for T.

Proof. Assume that (a) is satisfied. Since T is point dissipative, we can find a closed and bounded subset B_0 in (M,d) such that, for each $x \in M$, there exists $k = k(x) \geq 0$, $T^n(x) \in B_0$ $\forall n \geq k$. Define

$$J(B_0) := \{y \in B_0 : T^n(y) \in B_0 \ \forall n \geq 0\}.$$

Thus, $T(J(B_0)) \subset J(B_0)$, and for every $x \in M$, there exists $k = k(x) \geq 0$ such that $T^k(x) \in J(B_0)$. Since $J(B_0)$ is closed and bounded, and T is asymptotically smooth, Lemma 2.2 implies that $\omega(J(B_0))$ is compact invariant and attracts points of M.

Assume, in addition, that (b) is satisfied. We claim that there exists an $\varepsilon > 0$ such that $\gamma^+(N(\omega(J(B_0)),\varepsilon))$ is bounded. Assume, by contradiction, that $\gamma^+(N(\omega(J(B_0)), \frac{1}{n+1}))$ is unbounded for each $n > 0$. Let $z \in M$ be fixed. Then we can find a sequence $x_n \in N(\omega(J(B_0)), \frac{1}{n+1})$, and a sequence of integers $m_n \geq 0$ such that $d(z, T^{m_n}(x_n)) \geq n$. Since $\omega(J(B_0))$ is compact, we can always assume that $x_n \to x \in \omega(J(B))$ as $n \to +\infty$. Since $H := \{x_n : n \geq 0\} \cup \{x\}$ is compact, assumption (b) implies that $\gamma^+(H)$ is bounded, which is a contradiction. Let $D = \overline{\gamma^+(N(\omega(J(B_0)),\varepsilon))}$. Then D is closed, bounded, and positively invariant for T. Since $\omega(J(B_0))$ attracts points of M for T, and $\omega(J(B_0)) \subset N(\omega(J(B_0)),\varepsilon) \subset \operatorname{int}(D)$, we deduce that, for each $x \in M$, there exists $k = k(x) \geq 0$ such that $T^k(x) \in \operatorname{int}(D)$. It then follows that, for each compact subset C of M, there exists an integer $k \geq 0$ such that $T^k(C) \subset D$. Thus, the set $A := \omega(D)$ attracts every compact subset of M. Fix a bounded neighborhood V of A. By Lemma 2.4, it follows that A is stable, and hence, there is a neighborhood W of A such that $T^n(W) \subset V$ $\forall n \geq 0$. Clearly, the set $U := \cup_{n\geq 0} T^n(W)$ is a bounded neighborhood of A, and $T(\overline{U}) \subset \overline{U}$. Since T is asymptotically smooth, there is a compact set $J \subset \overline{U}$ such that J attracts $\overline{U}$. By Lemma 2.2, $\omega(\overline{U})$ is nonempty, compact, invariant for T, and attracts $\overline{U}$. Since A attracts $\omega(\overline{U})$, then $\omega(\overline{U}) \subset A$. Thus, A is a global attractor for T.

To prove the last part of the theorem, without loss of generality we assume that B is a bounded subset of M and $\gamma^+(B)$ is bounded. We set $K = \overline{\gamma^+(B)}$. Then $T(K) \subset K$. Since K is bounded and T is asymptotically smooth, there exists a compact C which attracts K for T. Note that $T^k(B) \subset T^k(\gamma^+(B)) \subset T^k(K)$ $\forall k \geq 0$. Thus, C attracts B for T. By Lemma 2.2, we deduce that $\omega(B)$ is nonempty, compact, invariant for T, and attracts B. Since A is a global attractor for T, it follows that A attracts compact subsets of M. By the invariance of $\omega(B)$ for T, we deduce that $\omega(B) \subset A$, and hence, A attracts B for T. □

Remark 2.7. From the first part of the proof of Theorem 2.6, it is easy to see that if T is point dissipative and asymptotically smooth, then there exists a nonempty, compact, and invariant subset C of M for T such that C attracts every point in M for T.

The following lemma provides sufficient conditions for the positive orbit of a compact set to be bounded.

LEMMA 2.8. *Assume that T is point dissipative. If C is a compact subset of M with the property that, for every bounded sequence $\{x_n\}_{n\geq 0}$ in $\gamma^+(C)$, $\{x_n\}_{n\geq 0}$ or $\{T(x_n)\}_{n\geq 0}$ has a convergent subsequence, then $\gamma^+(C)$ is bounded in M.*

Proof. Since T is point dissipative, we can choose a bounded and open subset V of M such that for each $x \in M$ there exists $n_0 = n_0(x) \geq 0$ such that $T^n(x) \in V$ $\forall n \geq n_0$. By the continuity of T and the compactness of C, it follows that there exists a positive integer $r = r(C)$ such that for any $x \in C$ there exists an integer $k = k(x) \leq r$ such that $T^k(x) \in V$. Let $z \in M$ be fixed. Assume, by contradiction, that $\gamma^+(C)$ is unbounded. Then there exists a sequence $\{x_p\}$ in $\gamma^+(C)$ such that

$$x_p = T^{m_p}(z_p),\ z_p \in C,\ \text{and}\ \lim_{p\to\infty} d(z, x_p) = \infty.$$

Since T is continuous and C is compact, without loss of generality we can assume that

$$\lim_{p\to\infty} m_p = \infty,\ \text{and}\ m_p > r,\ x_p \notin V\ \forall p \geq 1.$$

For each $z_p \in C$, there exists an integer $k_p \leq r$ such that $T^{k_p}(z_p) \in V$. Since $x_p = T^{m_p}(z_p) \notin V$, there exists an integer $n_p \in [k_p, m_p)$ such that

$$y_p = T^{n_p}(z_p) \in V,\ \text{and}\ T^l(y_p) \notin V\ \forall 1 \leq l \leq l_p = m_p - n_p.$$

Clearly, $x_p = T^{l_p}(y_p)\ \forall p \geq 1$, and $\{y_p\}$ is a bounded sequence in $\gamma^+(C)$.

We consider only the case where $\{y_p\}$ has a convergent subsequence, since the proof for the case where $\{T(y_p)\}$ has a convergent subsequence is similar. Thus, without loss of generality we can assume that $\lim_{p\to\infty} y_p = y \in \overline{V}$. In the case where the sequence $\{l_p\}$ is bounded, there exist an integer $\hat{l}$ and sequence $p_k \to \infty$ such that $l_{p_k} = \hat{l}\ \forall k \geq 1$, and hence,

$$d(z, T^{\hat{l}}(y)) = \lim_{k\to\infty} d(z, T^{\hat{l}}(y_{p_k})) = \lim_{k\to\infty} d(z, x_{p_k}) = \infty,$$

which is impossible. In the case where the sequence $\{l_p\}$ is unbounded, there exists a subsequence $l_{p_k} \to \infty$ as $k \to \infty$. Then for each fixed $m \geq 1$ there exists an integer k_m such that $m \leq l_{p_k}\ \forall k \geq k_m$, and hence,

$$T^m(y_{p_k}) \in M \setminus V\ \forall k \geq k_m.$$

Letting $k \to \infty$, we obtain

$$T^m(y) \in M \setminus V\ \forall m \geq 1,$$

which contradicts the definition of V. $\square$

The following result on the existence of strong global attractors is implied by [8, Theorems 3.1 and 3.4]. Since the proof of this result was not provided in [8], we include a simple proof of it.

THEOREM 2.9. *Let T be a continuous map on a complete metric space (M, d). Assume that T is point dissipative on M, and one of the following conditions holds:*

(a) *T^{n_0} is compact for some integer $n_0 \geq 1$, or*

(b) *T is asymptotically smooth and, for each bounded set $B \subset M$, there exists $k = k(B) \geq 0$ such that $\gamma^+(T^k(B))$ is bounded.*

Then there is a strong global attractor A for T.

Proof. The conclusion in case (b) is an immediate consequence of Theorem 2.6. In the case of (a), since T^{n_0} is compact for some integer $n_0 \geq 1$, it suffices to show that for each compact subset $C \subset M$, $\cup_{n\geq 0} T^n(C)$ is bounded. By applying Lemma 2.8 to $\widetilde{T} = T^{n_0}$, we deduce that for each compact subset $C \subset M$, $\cup_{n\geq 0}\widetilde{T}^n(C)$ is bounded. So Theorem 2.6 implies that $\widetilde{T}$ has a global attractor $\widetilde{A} \subset M$. We set $\widetilde{B} = \cup_{0\leq k\leq n_0-1} T^k(\widetilde{A})$. By the continuity of T, it then follows that $\widetilde{B}$ is compact and attracts every compact subset of M for T, and hence, the result follows from Theorem 2.6. $\square$

Remark 2.10. It is easy to see that a metric space (M, d) is complete if and only if for any subset B of M, $\kappa(B) = 0$ implies that $\overline{B}$ is compact. However, we can prove that Lemmas 2.2 and 2.4 also hold for noncomplete metric spaces by employing the equivalence between the compactness and the sequential compactness for metric spaces. It then follows that Theorems 2.6 and 2.9 are still valid for any metric space. We refer to [3, 20] for the existence of strong global attractors of continuous-time semiflows on a metric space.

3. Persistence and attractors. Let (M, d) be a complete metric space, and let $\rho : M \to [0, +\infty)$ be a continuous function. We define

$$M_0 := \{x \in M : \rho(x) > 0\} \text{ and } \partial M_0 := \{x \in M : \rho(x) = 0\}.$$

A subset $B \subset M_0$ is said to be *ρ-strongly bounded* if B is bounded in (M, d) and $\inf_{x \in B} \rho(x) > 0$. Throughout this section, we always assume that $T : M \to M$ is a continuous map with $T(M_0) \subset M_0$.

DEFINITION 3.1. *T is said to be* ρ-uniformly persistent *if there exists $\varepsilon > 0$ such that* $\liminf_{n \to +\infty} \rho(T^n(x)) \geq \varepsilon$, $\forall x \in M_0$*;* weakly ρ-uniformly persistent *if there exists $\varepsilon > 0$ such that* $\lim_{n \to +\infty} \sup \rho(T^n(x)) \geq \varepsilon \; \forall x \in M_0$. *The set ∂M_0 is said to be* ρ-ejective for T *if there exists $\varepsilon > 0$ such that for every $x \in M$ with $0 < \rho(x) < \varepsilon$ there is $n_0 = n_0(x) \geq 0$ such that $\rho(T^{n_0}(x)) \geq \varepsilon$.*

For a given open subset $M_0 \subset M$, let $\partial M_0 := M \setminus M_0$. Then we can use the continuous function $\rho : M \to [0, \infty)$, defined by $\rho(x) = d(x, \partial M_0) \; \forall x \in M$, to obtain the traditional definition of persistence.

PROPOSITION 3.2. *Assume that there is a compact subset C of M that attracts every point in M for T. Then the following statements are equivalent:*

(1) *T is weakly ρ-uniformly persistent.*

(2) *T is ρ-uniformly persistent.*

(3) *∂M_0 is ρ-ejective for T.*

Proof. The observations (1)$\Leftrightarrow$(3) and (2)$\Rightarrow$(1) are obvious. Let us prove that (1)$\Rightarrow$(2). Let $\varepsilon > 0$ be fixed such that

$$\lim_{n \to +\infty} \sup \rho(T^n(x)) \geq \varepsilon \; \forall x \in M_0. \tag{1}$$

Then for each $x \in M_0$, and each $n \geq 0$, there exists $p \geq 0$ such that $\rho(T^{n+p}(x)) \geq \varepsilon/2$. Assume that T is not ρ-uniformly persistent. Then we can find a sequence $\{x_m\}_{m \geq 0} \subset M_0$ such that

$$\lim_{n \to +\infty} \inf \rho(T^n(x_m)) \leq \frac{1}{m+1} \; \forall m \geq 0.$$

So there exist $l_m \geq 1$ and $n_m \geq 0$ such that

$$\begin{aligned} &d(T^{n_m}(x_m), C) \leq \frac{1}{m+1}, \; \rho(T^{n_m}(x_m)) \geq \varepsilon/2, \\ &\rho\left(T^{n_m+k}(x_m)\right) \leq \varepsilon/2 \, \forall \; k = 1, \ldots, l_m, \text{ and} \\ &\rho\left(T^{n_m+l_m}(x_m)\right) \leq \frac{1}{m+1}. \end{aligned}$$

Since C is compact, by taking a subsequence that we denote with the same index, we can always assume that $y_m = T^{n_m}(x_m) \to y \in C$. Since ρ and T are continuous, we deduce that

$$\rho(y) \geq \varepsilon/2, \text{ and } \rho\left(T^k(y)\right) \leq \varepsilon/2 \;\; \forall k = 1, \ldots, l,$$

where $l = \lim_{m\to+\infty} \inf l_m$. If $l < +\infty$, we have $\rho\left(T^l(y)\right) = 0$, which is impossible because $T(M_0) \subset M_0$. If $l = +\infty$, we have

$$\lim_{n\to+\infty} \sup \rho\left(T^n(y)\right) \le \varepsilon/2 < \varepsilon,$$

which contradicts (1). ∎

We note that the concept of general ρ-persistence was used in [27, 23, 30]. It was also shown in [27] that the ρ-uniform persistence implies the weak ρ-uniform persistence for nonautonomous semiflows under appropriate conditions. The following result shows that the notion of ρ-uniform persistence is independent of the choice of continuous function ρ.

PROPOSITION 3.3. *Let $\xi : M \to [0, +\infty)$ be a continuous function such that $\partial M_0 = \{x \in M : \xi(x) = 0\}$. Assume that there is a compact subset of M that attracts every point in M. Then T is ρ-uniformly persistent if and only if T is ξ-uniformly persistent.*

Proof. It suffices to prove that ρ-uniform persistence implies ξ-uniform persistence since the problem is symmetric. Let us first remark that T is ρ-uniformly persistent if and only if there exists $\varepsilon > 0$ such that

$$\inf_{x\in M_0} \inf_{y\in\omega(x)} \rho(y) \ge \varepsilon,$$

where $\omega(x)$ is the omega-limit set of the positive orbit of x. Define

$$A_\omega = \cup_{x\in M_0}\omega(x) \text{ and } V = \{y \in M : \rho(y) \ge \varepsilon\}.$$

Then

$$\inf_{x\in M_0} \inf_{y\in\omega(x)} \rho(y) = \inf_{x\in A_\omega} \rho(x) \ge \varepsilon.$$

Clearly, $A_\omega \subset C$, so $\overline{A}_\omega$ is compact. Since A_ω is included in $V \subset M_0$, which is closed, we deduce that $\overline{A}_\omega \subset V \cap C \subset M_0$. So $\overline{A}_\omega \subset M_0$ is compact, and hence, there exists $\eta > 0$ such that $\inf_{x\in\overline{A_\omega}} \xi(x) \ge \eta$, which implies that T is ξ-uniformly persistent. ∎

Let A be a nonempty subset of M. A is said to be *ejective for T* if there exists a neighborhood V of A such that for every $x \in (M \setminus A) \cap V$ there is $n_0 = n_0(x) \ge 0$ such that $T^{n_0}(x) \in M \setminus V$.

PROPOSITION 3.4. *Assume that $\partial M_0 \ne \emptyset$ and that there is a compact subset C of M which attracts every point in M for T. Then the following statements are equivalent:*

(1) *T is ρ-uniformly persistent.*

(2) *∂M_0 is ejective for T.*

Proof. Assume first that (1) is true. Let $\varepsilon > 0$ be fixed such that

$$\lim_{n\to+\infty} \sup \rho\left(T^n(x)\right) \ge \varepsilon \ \forall x \in M_0.$$

Then it is clear that ∂M_0 is ejective for T, with $V = \{x \in M : \rho(x) \le \varepsilon/2\}$.

Conversely, assume that ∂M_0 is ejective for T. Let V be a neighborhood of ∂M_0 such that for every $x \in M_0 \cap V$ there is $n_0 = n_0(x) \ge 0$ such that $T^{n_0}(x) \in M \setminus V$. By Proposition 3.3, it is sufficient to prove that T is ρ-uniformly persistent when

$\rho(x) = d(x, \partial M_0)$. Assume, by contradiction, that T is not ρ-uniformly persistent. Then for each $n \geq 1$ there exists $x_n \in M_0$, such that

$$\lim_{m\to+\infty} \sup \rho\left(T^m(x_n)\right) \leq \frac{1}{n}.$$

By the attractivity of C, it follows that for each $n \geq 1$ there exists $l_n \geq 0$ such that each $y_n := T^{l_n}(x_n) \in M_0$ satisfies

$$d\left(T^k\left(y_n\right), C\right) \leq \frac{2}{n} \text{ and } d\left(T^k\left(y_n\right), \partial M_0\right) \leq \frac{2}{n}\ \forall k \geq 0.$$

Since C is compact and V is a neighborhood of ∂M_0, there exists $\delta > 0$ such that

$$\{x \in M : d(x, C) \leq \delta \text{ and } d(x, \partial M_0) \leq \delta\} \subset V.$$

Let $n_0 \geq 2/\delta$ be fixed. Then we have $y_{n_0} \in M_0$, and

$$d\left(T^k\left(y_{n_0}\right), C\right) \leq \delta \text{ and } d\left(T^k\left(y_n\right), \partial M_0\right) \leq \delta\ \forall k \geq 0.$$

Thus, we obtain

$$y_{n_0} \in M_0 \cap V \text{ and } T^k\left(y_{n_0}\right) \in V\ \forall k \geq 0,$$

which is a contradiction. □

Observe that M_0 is an open subset in (M, d). In order to make M_0 become a complete metric space, we define a new metric function d_0 on M_0 by

$$d_0(x, y) = \left|\frac{1}{\rho(x)} - \frac{1}{\rho(y)}\right| + d(x, y)\ \forall x, y \in M_0. \tag{2}$$

Lemma 3.5. *(M_0, d_0) is a complete metric space.*

Proof. It is easy to see that d_0 is a metric function. Let $\{x_n\}_{n\geq 0}$ be a Cauchy sequence in (M_0, d_0). Since $d(x, y) \leq d_0(x, y)\ \forall x, y \in M_0$, we deduce that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (M, d), and there exists $x \in M$, such that $d(x_n, x) \to 0$ as $n \to +\infty$. To prove that $d_0(x_n, x) \to 0$ as $n \to +\infty$, it is sufficient to show that $x \in M_0$. Given $\varepsilon > 0$, since $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in (M_0, d_0), there exists $n_0 \geq 0$ such that $d_0(x_n, x_p) \leq \varepsilon\ \forall n, p \geq n_0$. In particular, we have $d_0(x_n, x_{n_0}) \leq \varepsilon\ \forall n \geq n_0$. Then

$$\left|\frac{1}{\rho(x_n)} - \frac{1}{\rho(x_{n_0})}\right| \leq \varepsilon\ \forall n \geq n_0.$$

So there exists $r > 0$ such that $\inf_{n\geq 0} \rho(x_n) \geq r$. Since ρ is continuous and $d(x_n, x) \to 0$ as $n \to +\infty$, we deduce that $\rho(x) \geq r$, and hence $x \in M_0$. Thus, (M_0, d_0) is complete. □

For any two subsets $A, B \subset M$, we denote

$$\delta(B, A) = \sup_{x\in B} \inf_{y\in A} d(x, y),$$

and if $A, B \subset M_0$, we denote

$$\delta_0(B, A) = \sup_{x\in B} \inf_{y\in A} d_0(x, y).$$

Lemma 3.6. *The following two statements are valid:*

(1) *Let* $\{B_t\}_{t\in I}$ *be a family of subsets of* M_0, *where* I *is a unbounded subset of* $[0,+\infty)$. *If* $A \subset M_0$ *is compact in* (M,d) *and* $\lim_{t\to\infty} \delta(B_t, A) = 0$, *then* $\lim_{t\to\infty} \delta_0(B_t, A) = 0$.

(2) *If* T *is asymptotically smooth, then* T *is asymptotically smooth in* (M_0, d_0).

Proof. (1) We denote $k := \frac{1}{2}\inf_{x\in A}\rho(x) > 0$. Assume, by contradiction, that $\lim_{t\to+\infty}\sup \delta_0(B_t, A) > \varepsilon > 0$. Then we can find a sequence $\{t_p\}_{p\geq 0} \subset I$ such that $t_p \to +\infty$, $p \to +\infty$, and a sequence $\left\{x_{t_p}\right\}_{p\geq 0} \subset M_0$ such that $x_{t_p} \in B_{t_p}$, $d_0\left(x_{t_p}, A\right) \geq \varepsilon$ $\forall p \geq 0$. Since $d\left(x_{t_p}, A\right) \to 0$, as $p \to +\infty$, without loss of generality we can assume that there exists $x \in A$ such that $d(x_{t_p}, x) \to 0$, as $p \to +\infty$. Since ρ is continuous and $\rho(x) > k$, there exists $p_0 \geq 0$ such that $\rho(x_{t_p}) \geq k$ $\forall p \geq p_0$. Thus, we have

$$0 < \varepsilon \leq d_0\left(x_{t_p}, x\right) \leq k^{-2}\left|\rho(x_{t_p}) - \rho(x)\right| + d(x_{t_p}, x) \to 0 \text{ as } p \to +\infty,$$

which is a contradiction.

(2) It is easy to see that $T : (M_0, d_0) \to (M_0, d_0)$ is continuous. Let B be a bounded subset in (M_0, d_0) such that $T(B) \subset B$. Since T is asymptotically smooth, there exists a compact subset $C \subset M$ that attracts B for T. So $C_0 = C \cap \overline{B} \subset M_0$ is compact and attracts B for T. It easily follows that C_0 is also compact in (M_0, d_0). Since C_0 attracts B for T, statement (1) implies that C_0 attracts B for $T : (M_0, d_0) \to (M_0, d_0)$. $\square$

The main result of this section is the following theorem.

THEOREM 3.7. *Assume that* T *is asymptotically smooth and* ρ*-uniformly persistent, and that* T *has a global attractor* A. *Then* $T : (M_0, d) \to (M_0, d)$ *has a global attractor* A_0. *Moreover, for each subset* B *of* M_0, *if there exists* $k \geq 0$ *such that* $\gamma^+\left(T^k(B)\right)$ *is* ρ*-strongly bounded, then* A_0 *attracts* B *for* T.

Proof. We consider the continuous map $T : (M_0, d_0) \to (M_0, d_0)$. Since T is point dissipative and ρ-uniformly persistent, T is point dissipative in (M_0, d_0). Moreover, Lemma 3.6 implies that T is asymptotically smooth in (M_0, d_0). Let C be a compact subset in (M_0, d_0), and $\{x_p\}$ a bounded sequence in $\gamma^+(C)$ in (M_0, d_0). Then $x_p = T^{m_p}(z_p)$, $z_p \in C$ $\forall p \geq 1$, and the sequence $\{x_p\}$ is ρ-strongly bounded in (M, d). Since C is also compact in (M, d), we have $\lim_{m\to\infty}\delta(T^m(C), A) = 0$. Thus, $\{x_p\}$ has a convergent subsequence $x_{p_k} \to x$ in (M, d) as $k \to \infty$. By the continuity of ρ and the ρ-strong boundedness of $\{x_p\}$, it follows that $\rho(x) > 0$, i.e., $x \in M_0$, and hence, $x_{p_k} \to x$ in (M_0, d_0) as $k \to \infty$. Thus, Lemma 2.8 implies that positive orbits of compact sets are bounded for $T : (M_0, d_0) \to (M_0, d_0)$. Then the conclusion for $T : (M_0, d) \to (M_0, d)$ follows from Theorem 2.6, as applied to $T : (M_0, d_0) \to (M_0, d_0)$. $\square$

THEOREM 3.8. *Assume that* T *is point dissipative on* M *and* ρ*-uniformly persistent, and that one of the following conditions holds:*

(a) *There exists some integer* $n_0 \geq 1$ *such that* T^{n_0} *is compact on* M, *and* T^{n_0} *maps* ρ*-strongly bounded subsets of* M_0 *onto* ρ*-strongly bounded sets in* M_0, *or*

(b) T *is asymptotically smooth on* M, *and for every* ρ*-strongly bounded subset* $B \subset M_0$, *there exists* $k = k(B) \geq 0$ *such that* $\gamma^+\left(T^k(B)\right)$ *is* ρ*-strongly bounded in* M_0.

Then $T : (M_0, d) \to (M_0, d)$ *has a global attractor* A_0, *and* A_0 *attracts every* ρ*-strongly bounded subset in* M_0 *for* T.

Proof. Clearly, $T : (M_0, d_0) \to (M_0, d_0)$ is point dissipative. It is easy to see that condition (a) implies that $T^{n_0} : (M_0, d_0) \to (M_0, d_0)$ is compact, and that condition (b) implies that condition (b) of Theorem 2.9 holds for $T : (M_0, d_0) \to (M_0, d_0)$.

By Theorem 2.9, there is a strong global attractor A_0 for $T : (M_0, d_0) \to (M_0, d_0)$. Consequently, A_0 is a global attractor for $T : (M_0, d) \to (M_0, d)$, and A_0 attracts every ρ-strongly bounded subset in M_0 for T. $\square$

Remark 3.9. A result similar to Theorem 3.8 was already presented for discrete- and continuous-time dynamical systems in [29] and [10], respectively. The only difference, compared with the earlier results, is that we add a ρ-boundedness assumption for case (a). In fact, this assumption is necessary for the existence of a strong global attractor in M_0 for T (see two examples in section 5.1).

Remark 3.10. A family of mappings $\Phi(t) : M \to M$, $t \geq 0$, is called a continuous-time semiflow if $(x, t) \to \Phi(t)x$ is continuous, $\Phi(0) = Id$, and $\Phi(t) \circ \Phi(s) = \Phi(t + s)$ for $t, s \geq 0$. By similar arguments we can prove the analogues of Theorems 3.7 and 3.8 for a continuous-time semiflow $\Phi(t)$ on M with $\Phi(t)(M_0) \subset M_0$ $\forall t \geq 0$.

4. Coexistence steady states. In this section, we establish the existence of a coexistence steady state (i.e., the fixed point in M_0) for uniformly persistent dynamical systems.

Throughout this section we always assume that M is a closed and convex subset of a Banach space $(X, \|\cdot\|)$, that $\rho : M \to [0, +\infty)$ is a continuous function such that $M_0 = \{x \in M : \rho(x) > 0\}$ is nonempty and convex, and that $T : M \to M$ is a continuous map with $T(M_0) \subset M_0$. For convenience, we set $\partial M_0 := M \setminus M_0$.

Assume that $T : M_0 \to M_0$ has a global attractor A_0. By Definition 2.5, it easily follows that for every compact set $K \subset M_0$ there exists an open neighborhood of K which is attracted by A_0. This property of A_0 is enough to support the arguments in the proof of [29, Theorem 2.3] (see also [30, Theorem 1.3.6]) instead of the property that A_0 attracts ρ-strongly bounded sets in M_0. Thus, the proof of [29, Theorem 2.3] actually implies the following fixed point theorem.

THEOREM 4.1. *Assume that T is κ-condensing. If $T : M_0 \to M_0$ has a global attractor A_0, then T has a fixed point $x_0 \in A_0$.*

Note that a fixed point theorem for κ-condensing maps in [9] was used in the proof of [29, Theorem 2.3]. To generalize Theorem 4.1 to another class of maps, we need the following fixed point theorem, which is a combination of Theorems 3 and 5 in [9] (see also [7, Lemma 2.6.5]).

LEMMA 4.2 (Hale–Lopes fixed point theorem). *Assume that $K \subset B \subset S$ are convex subsets of a Banach space X, with K compact, S closed and bounded, and B open in S. If $T : S \to X$ is continuous, $T^n B \subset S$ $\forall n \geq 0$, and K attracts compact subsets of B, then there exists a closed bounded and convex subset $C \subset S$ such that $C = \overline{co}\left(\cup_{j \geq 1} T^j (B \cap C)\right)$. Moreover, if C is compact, then T has a fixed point in B.*

We should point out that in the above fixed point theorem the claim that T has a fixed point in B follows from the proof of [7, Lemma 2.6.5], where Horn's fixed point theorem [12] was used.

Motivated by Lemma 4.2 and the Poincaré maps associated with age-structured population models, we give the following definition.

DEFINITION 4.3. *Let M be a closed and convex subset of a Banach space X, and let $T : M \to M$ be a continuous map. Define $\widehat{T}(B) = \overline{co}\,(T(B))$ for each $B \subset M$. T is said to be convex κ-contracting if $\lim_{n\to\infty} \kappa(\widehat{T}^n(B)) = 0$ for each bounded subset $B \subset M$.*

Now we are ready to generalize Theorem 4.1 to convex κ-contracting maps.

THEOREM 4.4. *Assume that T is convex κ-contracting. If $T : M_0 \to M_0$ has a global attractor A_0, then T has a fixed point $x_0 \in A_0$.*

Proof. Since A_0 is a global attractor for $T : M_0 \to M_0$, the proof of [29, Theorem

2.3] (see also [30, Theorem 1.3.6]) implies that there are three convex subsets, $K \subset B \subset S \subset M$, such that $K \subset M_0$, $B \subset M_0$, and the assumptions of Lemma 4.2 hold for T. Let C be defined in Lemma 4.2. Define $\widehat{C} := \cup_{j\geq 1} T^j(B \cap C)$. Then we have

$$\widehat{C} = T(B \cap C) \cup T\left(\widehat{C}\right) \text{ and } C = \overline{co}\left(\widehat{C}\right),$$

and hence, $\widehat{C} \subset T(C)$. Thus, we further obtain

$$C \subset \widehat{T}(C) \subset \widehat{T}^2(C) \subset \cdots \subset \widehat{T}^n(C) \ \forall n \geq 0.$$

Since T is convex κ-contracting, it follows that $\kappa(C) \leq \kappa(\widehat{T}^n(C)) \to 0$ as $n \to \infty$. Then $\kappa(C) = 0$, and hence, C is compact. Now Lemma 4.2 implies the existence of a fixed point of T in A_0. ∎

Combining Theorems 2.6, 2.9, 3.7, 4.1, and 4.4 we have the following result on the existence of coexistence steady states for uniformly persistent systems, which is a generalization of [29, Theorem 2.3].

THEOREM 4.5. *Assume that*

(1) *T is point dissipative and ρ-uniformly persistent.*

(2) *One of the following two conditions holds:*

 (2a) *T^{n_0} is compact for some integer $n_0 \geq 1$, or*

 (2b) *for each compact subset $C \subset M$, $\gamma^+(C)$ is bounded.*

(3) *Either T is κ-condensing or T is convex κ-contracting.*

Then $T : M_0 \to M_0$ admits a global attractor A_0, and T has a fixed point in A_0.

Let $A \subset M$ and $B \subset M \setminus A$. A is said to be *ejective for T in B* if there exists a neighborhood V of A such that for each $x \in V \cap B$ there exists $n = n(x) \geq 0$ such that $T^n(x) \in M \setminus V$. A is said to be *ejective for T* if A is ejective for T in $M \setminus A$.

The following corollary is a generalization of [15, Theorem 4.1] on semi-ejective fixed points.

COROLLARY 4.6. *Assume that $T(\partial M_0) \subset \partial M_0$ and that there exists $\overline{x}_\partial \in \partial M_0$, a fixed point of T, which is globally asymptotically stable for $T : \partial M_0 \to \partial M_0$. Assume, in addition, that*

(1) *T is point dissipative and $\overline{x}_\partial$ is ejective for T in M_0.*

(2) *One of the following two conditions holds:*

 (2a) *T^{n_0} is compact for some integer $n_0 \geq 1$, or*

 (2b) *positive orbits of compact subsets of M are bounded.*

(3) *Either T is κ-condensing or convex κ-contracting.*

Then $T : M_0 \to M_0$ admits a global attractor A_0, and T has a fixed point in A_0.

Proof. By [29, Theorem 2.2] (see also [30, Theorem 1.3.1]), we deduce that T is ρ-uniformly persistent with $\rho(x) = d(x, \partial M_0)$. Now Theorem 4.5 completes the proof. ∎

We remark that when $\partial M_0 = \{\overline{x}_\partial\}$ in Corollary 4.6, we obtain a generalization of the classical Browder [2] ejective fixed point theorem.

A point $e \in M$ is said to be an equilibrium of a continuous-time semiflow $\Phi(t)$ on M if $\Phi(t)e = e \ \forall\ t \geq 0$. As a consequence of Theorems 4.1 and 4.4 and the proof of [29, Theorem 2.4] (see also [30, Theorem 1.3.7]), we have the following result on the existence of equilibrium in M_0 for $\Phi(t)$.

THEOREM 4.7. *Let $\Phi(t)$ be a continuous-time semiflow on M with $\Phi(t)(M_0) \subset M_0 \ \forall\ t \geq 0$. Assume that either $\Phi(t)$ is κ-condensing for each $t > 0$, or $\Phi(t)$ is convex κ-contracting for each $t > 0$, and that $\Phi(t) : M_0 \to M_0$ has a global attractor A_0. Then $\Phi(t)$ has an equilibrium $x_0 \in A_0$.*

In the rest of this section, we establish sufficient conditions for T to be convex κ-contracting.

LEMMA 4.8. *Let M be a closed and convex subset of a Banach space X, and let $T : M \to M$ be a continuous map which takes bounded sets to bounded sets. Assume that there exists a sequence of bounded linear operators $\{P_k\}_{k\geq 1} \in \mathcal{L}(X,X)$ such that*

(1) *for each bounded subset $B \subset M$, $(Id - P_1)T(B)$ is relatively compact;*
(2) *one of the following conditions holds:*
   (2a) *There exists $n_0 \geq 0$ such that P_{n_0} is compact, and if $k \in \{1,\dots,n_0-1\}$, $C \subset M$, and $(Id - P_k)C$ is compact, then $(Id - P_{k+1})\,T(C)$ is compact.*
   (2b) *There exists $c \in (0,1)$ such that $\|P_{k+1}T(x)\| \leq c\,\|P_k x\|$ $\forall\, x \in M$, $\forall\, k \geq 1$, and if $k \geq 1$, $C \subset M$, and $(Id - P_k)C$ is compact, then $(Id - P_{k+1})\,T(C)$ is compact.*

Then T is convex κ-contracting.

Proof. Let $B \subset M$ be a bounded subset of M. Since $(Id - P_1)T(B)$ is relatively compact and P_1 is linear, it follows that

$$(I - P_1)\overline{co}(T(B)) = \overline{co}((I - P_1)T(B)) \text{ is compact,}$$

$$\text{and } (Id - P_1)\overline{co}(T(B)) \text{ is compact.}$$

Thus, $(Id - P_2)\overline{co}(T(\overline{co}(T(B))))$ is compact, and, by induction, $(Id - P_{k+1})\widehat{T}^k(B)$ is compact $\forall k \in \{1,\dots,n_0-1\}$ if (2a) holds, and $\forall\ k \geq 1$ if (2b) holds. If (2a) holds, since P_{n_0} is compact, we deduce that $\widehat{T}^{n_0}(B)$ is compact, and hence, $\kappa(\widehat{T}^n(B)) = 0$ $\forall n \geq n_0$. If (2b) holds, then the boundedness of linear operator P_1 implies that

$$\sup_{y\in\overline{co}(T(B))} \|P_1 y\| = \sup_{x\in T(B)} \|P_1 x\| \leq c \sup_{x\in B} \|x\| .$$

Similarly, we have

$$\sup_{y\in\overline{co}(T(\widehat{T}(B)))} \|P_2 y\| = \sup_{x\in T(\widehat{T}(B))} \|P_2 x\| \leq c \sup_{x\in\widehat{T}(B)} \|P_1 x\|$$
$$\leq c^2 \sup_{x\in B} \|x\| .$$

By induction, it follows that

$$\sup_{y\in\widehat{T}^k(B)} \|P_k y\| \leq c^k \sup_{x\in B} \|x\| \ \ \forall k \geq 1.$$

Let $\delta_k := c^k \sup_{x\in B}\|x\|$. Since $(Id - P_k)\widehat{T}^k(B)$ is compact, there exists $x_1,\dots,x_{m(k)} \in (Id - P_k)\widehat{T}^k(B)$ such that

$$(Id - P_k)\widehat{T}^k(B) \subset \cup_{j=1,\dots,m(k)} B_M\left(x_j,\delta_k\right),$$

where $B_M(x_j,\delta_k) = \{x \in M : \|x - x_j\| < \delta_k\}$. Thus, we have

$$(Id - P_k)\widehat{T}^k(B) + P_k\widehat{T}^k(B) \subset \cup_{j=1,\dots,m(k)} B_M\left(x_j, 2\delta_k\right).$$

Since $\widehat{T}^k(B) \subset (Id - P_k)\widehat{T}^k(B) + P_k\widehat{T}^k(B)$, it follows that

$$\kappa(\widehat{T}^k(B)) \leq \kappa\left((Id - P_k)\widehat{T}^k(B) + P_k\widehat{T}^k(B)\right) \leq 2\delta_k \to 0 \text{ as } k \to +\infty.$$

Thus, T is convex κ-contracting. $\square$

We complete this section with an example of convex κ-contracting maps.

Example 4.9. Consider $T : L^1_+(0,c) \to L^1_+(0,c)$, with $c \in (1,+\infty]$, defined by

$$T(\varphi)(a) = \begin{cases} \chi(\varphi)\varphi(a-1) \text{ if } 1 \le a < c, \\ \lambda \text{ if } a \in (0,1), \end{cases}$$

where $\lambda > 0$, and $\chi : L^1_+(0,c) \to [0,\alpha]$ (with $0 < \alpha$) is a continuous map. We choose for each integer $k \ge 1$, $P_k : L^1(0,c) \to L^1(0,c)$ the operator defined by

$$P_k(\varphi)(a) = \begin{cases} \varphi(a) \text{ if } a \in (0,c) \cap (k,+\infty), \\ 0 \text{ otherwise.} \end{cases}$$

If $c < +\infty$, then (2a) holds. If $c = +\infty$ and $\alpha < 1$, then (2b) holds. Thus, Lemma 4.8 implies that T is convex κ-contracting. Note that in this example we need to impose some additional conditions on χ to show that T is κ-condensing.

5. Five examples. In this section, we first provide four examples of discrete- and continuous-time semiflows which admit global attractors, but no strong global attractors, in the complete metric spaces (M_0, d_0) introduced in section 3. Then we give an example showing applicability of Theorem 4.5 in the case of a convex κ-contracting map. Our examples are highly motivated by age-structured population models. We refer to Webb [28], Iannelli [14], and Anita [1] for the classical approach and to Thieme [25] and Magal and Thieme [16] (and references therein) for the integrated semigroup approach to this class of evolutionary equations.

5.1. Asymptotically smooth semiflows on (M_0, d_0). Let $C([0,1],\mathbb{R})$ be endowed with the usual norm $\|\varphi\|_\infty = \sup_{a\in[0,1]} |\varphi(a)|$. Let $M := C_+([0,1],\mathbb{R})$ be endowed with the metric $d(x,y) = \|x-y\|$, and $T : M \to M$ be defined by

$$T(\varphi) = \delta \frac{\mathcal{F}_\beta(\varphi)}{1+\mathcal{F}_\beta(\varphi)} 1_{[0,1]},$$

where $1_{[0,1]}(a) = 1\ \forall a \in [0,1]$, and $\mathcal{F}_\beta(\varphi) = \int_0^1 \beta(a)\varphi(a)da\ \forall \varphi \in X$. We assume that

(A1) $\delta > 1, \beta \in C([0,1],\mathbb{R})$, $\int_0^1 \beta(a)da = 1$, $\beta(a) > 0\ \forall a \in [0,1)$, and $\beta(1) = 0$.

Consider the following discrete-time system on M:

$$u_{n+1} = T(u_n)\ \ \forall n \ge 0, \text{ and } u_0 \in M.$$

It is easy to see that the map T is continuous and maps bounded sets into compact sets of M. Note that $T(M) \subset [0,\delta]\,1_{[0,1]} = \{\alpha 1_{[0,1]} : \alpha \in [0,\delta]\}$ is bounded. So T is compact and point dissipative and has a strong global attractor in M. Set

$$\partial M_0 = \{0\}, \quad M_0 = M \setminus \{0\}, \quad \rho(x) = \|x\|_\infty.$$

Clearly, $T(M_0) \subset M_0$, $T(\partial M_0) \subset \partial M_0$, and the fixed points of T are 0 and $\overline{u} = (\delta-1)1_{[0,1]}$. Then it is easy to see that for each $\varphi \in M_0$, $T^m(\varphi) \to \overline{u}$, as $m \to +\infty$. So T is ρ-uniformly persistent. Let $\overline{\alpha} = (\delta - 1)$ and $B := \{x \in M : \|x\|_\infty = \overline{\alpha}\}$. Since $\beta(1) = 0$, we have $\mathcal{F}_\beta(B) = (0,\overline{\alpha}]$. Moreover, $T(B) = \{\alpha 1_{[0,1]} : \alpha \in (0,\overline{\alpha}]\}$, and $T^n(B) = T(B)\ \forall n \ge 1$. Thus, there exists no compact subset in M_0 that attracts B for T. In particular, there is no strong global attractor for $T : (M_0,d_0) \to (M_0,d_0)$, where d_0 is defined as in (2).

Next we consider the continuous-time semiflow $\{U(t)\}_{t\geq 0}$ on $M := L^1_+(0,1)$, which is generated by the following age-structured model:

$$(3) \qquad \begin{cases} \frac{\partial u(t)}{\partial t} + \frac{\partial u(t)}{\partial a} = -\mu(a)u(t)(a) - \mathcal{F}_\Gamma(u(t))\,u(t)(a), \quad a \in (0,1), \\ u(t,0) = \mathcal{F}_\beta(u(t)), \\ u(0) = \varphi \in L^1_+(0,1), \end{cases}$$

where for each $\chi \in L^\infty(0,1)$, and each $\varphi \in L^1(0,1)$, $\mathcal{F}_\chi(\varphi) = \int_0^1 \chi(a)\varphi(a)da$. We assume that

(A2) $\beta \in (0,+\infty)$, $\mu \in L^1_{loc}[0,1)$, $\mu \geq 0$, $\lim_{a\to 1^-} \int_0^a \mu(r)dr = +\infty$, $\int_0^1 \beta \exp(-\int_0^a \mu(s)ds)da > 1$, and

$$\Gamma(a) = \frac{1}{\int_0^1 \exp(-\int_0^s \mu(r) + \lambda_0 dr)ds} \int_a^1 \exp\left(-\int_a^s \mu(r) + \lambda_0 dr\right) ds \ \forall a \in [0,1],$$

where $\lambda_0 > 0$ is the unique solution of $\int_0^1 \beta \exp\left(-\int_0^a \mu(s) + \lambda_0 ds\right) da = 1$.

Let $\{T(t)\}_{t\geq 0}$ be the C_0-semigroup of bounded linear operators generated by $A : D(A) \subset L^1(0,1) \to L^1(0,1)$ with

$$A\varphi = -\frac{d\varphi}{da} - \mu\varphi \ \forall \ \varphi \in D(A),$$

$$D(A) = \left\{\varphi \in W^{1,1}(0,1) : \mu\varphi \in L^1(0,1) \text{ and } \varphi(0) = \int_0^1 \beta\varphi(a)da\right\}.$$

Let $P : L^1(0,1) \to L^1(0,1)$ be the bounded linear operator of projection defined by

$$P(\varphi)(a) = \int_0^1 \Gamma(s)\varphi(s)ds\chi(a) \ \forall \varphi \in L^1(0,1),$$

where $\chi(a) = \alpha \exp\left(-\int_0^a \mu(s) + \lambda_0 ds\right)$, and

$$\alpha = \left(\int_0^1 \Gamma(s) \exp\left(-\int_0^a \mu(s) + \lambda_0 ds\right) ds\right)^{-1}.$$

Then $PT(t) = T(t)P = e^{\lambda_0 t}P \ \forall t \geq 0$, and there exist $\delta > 0$ and $M \geq 1$ such that

$$\|(Id - P)T(t)\| \leq Me^{(\lambda_0 - \delta)t} \ \forall t \geq 0.$$

Moreover, we have

$$U(t)x = \frac{T(t)x}{1 + \int_0^t \mathcal{F}_\Gamma(T(s)x)\,ds} \ \forall t \geq 0, \forall x \in M.$$

It is easy to see that for each $\varphi \in M_0$, $U(t)\varphi \to \lambda_0\chi$, as $t \to +\infty$. Since $T(t)$ is compact for $t \geq 2$, $U(t)$ is compact for $t \geq 2$. So $\{U(t)\}_{t\geq 0}$ has a strong global attractor. Set

$$\partial M_0 = \{0\}, \quad M_0 = M \setminus \{0\}, \quad \rho(\varphi) = \|\varphi\|_{L^1(0,1)} \ \forall \varphi \in M.$$

Since $T(t)$ is irreducible, we have $U(t)(\partial M_0) \subset \partial M_0$, and $U(t)M_0 \subset M_0 \ \forall t \geq 0$. Since $U(t)\varphi \to \lambda_0\chi$, as $t \to +\infty$, we deduce that $U(t)$ is ρ-uniformly persistent. So $U(t) : (M_0,d) \to (M_0,d)$ has a global attractor. Let

$$B := \left\{\varphi \in L^1_+(0,1) : \|\varphi\|_{L^1(0,1)} = 1\right\}.$$

Then B is ρ-strongly bounded. Since $\Gamma(1) = 0$, we deduce that there exists $c > 0$ such that $(0, c] \subset \mathcal{F}_\Gamma(B)$. We further claim that for each $\varepsilon > 0$ and $t_0 > 0$, there exist $t_1 > t_0$ and $\varphi \in B$ such that $\|U(t_1)\varphi\|_{L^1(0,1)} < \varepsilon$. Indeed, given $\varepsilon > 0$ and $t_0 > 0$, we can choose $t_1 > t_0$ such that $Me^{-\delta t_1} \le \varepsilon/2$. Then for every $\varphi \in B$, we have

$$\|U(t_1)\varphi\| \le \frac{\mathcal{F}_\Gamma(\varphi)\|\chi\|}{\left[1 - \frac{\mathcal{F}_\Gamma(\varphi)}{\lambda_0}\right] e^{-\lambda_0 t_1} + \frac{\mathcal{F}_\Gamma(\varphi)}{\lambda_0}} + \frac{\varepsilon}{2},$$

and hence, by choosing $\varphi \in B$ with $\mathcal{F}_\Gamma(\varphi)$ small enough, we obtain $\|U(t_1)\varphi\| \le \varepsilon$. This claim shows that for each $t_0 > 0$, $\cup_{t \ge t_0} U(t)B$ is not ρ-strongly bounded. So there exists no compact set in M_0 that attracts B for $U(t)$. In particular, there exists no strong global attractor for the semiflow $U(t) : (M_0, d_0) \to (M_0, d_0)$, where d_0 is defined as in (2).

5.2. κ-contracting maps on (M_0, d_0). In this subsection, we construct κ-contracting maps on (M_0, d_0) such that they admit a global attractor, but no strong global attractor.

We set

$$X = L^1((0, +\infty), \mathbb{R}) \times \mathbb{R}, \; X_+ = L^1_+((0, +\infty), \mathbb{R}) \times \mathbb{R}_+$$

and endow X with the product norm $\|(\varphi, y)\| = \|\varphi\|_{L^1} + |y|$. Define $1_{[0,1]} \in X$ by $1_{[0,1]}(l) = 1 \; \forall l \in (0, 1)$, and $1_{[0,1]}(l) = 0 \; \forall l \in [1, \infty)$. Let a, b, and c be three real numbers. Define $T : X_+ \to X_+$ by $T(\varphi, y) = (T_1(\varphi, y), T_2(\varphi, y))$ with

$$T_1(\varphi, y) = a\varphi(\cdot + 1) + \left[a \int_0^1 \varphi(l)dl + c\frac{\int_0^1 \varphi(l)dl}{1 + \|(\varphi, y)\|}\right] 1_{[0,1]},$$
$$T_2(\varphi, y) = ay + b\frac{\|(\varphi, y)\|}{1 + \|(\varphi, y)\|}.$$

We assume that

(A3) $a \in (0, 1)$, $b > 0$, $c > 0$, $\sqrt{a} < a + b < 1$, and $a + c > 1$.

Consider the discrete-time system

$$x_{n+1} = T(x_n) \;\; \forall n \ge 0, \text{ and } x_0 \in X_+.$$

It is easy to see that $T^n(0, y) \to 0$, as $n \to +\infty$. Clearly, T is not uniformly persistent for $X_+ \setminus \{0\}$. We will find a closed subset M of X_+, such that it contains 0 and is positively invariant for T, and show that T is uniformly persistent for $M \setminus \{0\}$.

Lemma 5.1. *There exists a nondecreasing and right-continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0, f(x) > 0 \; \forall x > 0$, $\lim_{x \to 0} f(x) = 0$, and the set $M := \{(\varphi, y) \in X_+ : y \le f(\|\varphi\|)\}$ is positively invariant for T.*

Proof. We define $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ by

$$F(x_1, x_2) = \left(ax_1, ax_2 + b\frac{x_1 + x_2}{1 + x_1 + x_2}\right) \; \forall x = (x_1, x_2) \in \mathbb{R}^2_+.$$

Then F is nondecreasing on $\mathbb{R}^2_+$. Set

$$\chi(t) = (ta + (1 - t), 1) \;\; \forall t \in [0, 1].$$

By induction, we define $\chi : \mathbb{R}_+ \to \mathbb{R}_+^2$ by

$$\chi(t) = F(\chi(t-1)) \; \forall t \in (n, n+1], \;\; \forall n \geq 1.$$

Note that $\chi(1)_1 = F\left(\chi(0)\right)_1$ and $a < 1$. Then the function $t \to \chi(t)_1$ is strictly decreasing and continuous. Since $F\left(1,1\right) \leq \left(a,1\right)$, the function $t \to \chi(t)_2$ is nonincreasing and left-continuous. Moreover, since $a+b<1$, we have $\lim_{t\to+\infty} \chi(t) = 0$. We further set

$$\chi(t) = (1-t, 1) \;\; \forall t \in (-\infty, 0]\,.$$

Since $\chi(t)_1$ is strictly decreasing in $t \in \mathbb{R}$, we can define

$$f(x) = \begin{cases} \chi(\chi(x)_1^{-1})_2 \text{ if } x > 0, \\ 0 \text{ if } x = 0. \end{cases}$$

It is easy to see that f has the desired properties.

Let $D := \left\{(x_1, x_2) \in \mathbb{R}_+^2 : x_2 \leq f(x_1)\right\}$. Since f is nondecreasing and right-continuous, it easily follows that D is closed. Now we show that $F(D) \subset D$. Let $x = (x_1, x_2) \in D$; then $x_2 \leq f\left(x_1\right)$. If $x_1 = 0$, there is nothing to prove because $F(0) = 0$. Assume that $x_1 > 0$; then there exists $t \in \mathbb{R}$ such that $\chi(t)_1 = x_1$, and hence, $x_2 \leq f(x_1) = \chi(t)_2$. Clearly, $x = (x_1, x_2) \leq \chi(t)$, and $F(x) \leq F(\chi(t))$. In the case where $t \geq 0$, we have

$$\chi(t+1)_1 = F\left(\chi(t)\right)_1 = F\left(x\right)_1,$$

and hence,

$$f(F(x)_1) = \chi(t+1)_2 = F\left(\chi(t)\right)_2 \geq F\left(x\right)_2,$$

which implies that $F(x) \in D$. In the case where $t \leq 0$, we have

$$x_1 \geq F\left(x\right)_1 = F\left(\chi(t)\right)_1 = a\chi(t)_1 = a\left(1-t\right) \geq a = \chi(1)_1,$$

and hence, there exists $s \in [t, 1]$ such that $\chi(s)_1 = F\left(x\right)_1$. It then follows that

$$f(F(x)_1) = \chi(s)_2 = 1 \geq F(\chi(t))_2 \geq F(x)_2,$$

which implies that $F(x) \in D$. This proves that $F(D) \subset D$.

Finally, we prove that $T(M) \subset M$. For any $(\varphi, y) \in M$, we have $(\|\varphi\|, y) \in D$, and hence, the positive invariance of D for F implies that $F(\|\varphi\|, y)_2 \leq f(F(\|\varphi\|, y)_1)$. Note that $\|T_1(\varphi, y)\| \geq a\|\varphi\| = F(\|\varphi\|, y)_1$ and $T_2(\varphi, y) = F(\|\varphi\|, y)_2$. By the monotonicity of f, it then follows that

$$T_2(\varphi, y) = F(\|\varphi\|, y)_2 \leq f(F(\|\varphi\|, y)_1) \leq f(\|T_1(\varphi, y)\|),$$

which implies that $T(\varphi, y) \in M$. Thus, M is positively invariant for T. $\square$

Now we consider $T : M \to M$, where M is endowed with the usual distance $d(x, \widehat{x}) = \|x - \widehat{x}\|$. We set

$$\partial M_0 = \{0\}\,, \;\; M_0 = M \setminus \{0\}\,, \;\text{ and } \rho(x) = \|x\|\,.$$

Since T is the sum of a compact operator and a linear operator with norm being a, we have $\kappa(T(B)) \le a\kappa(B)$ for any bounded set $B \subset M$. Thus, T is a κ-contraction. Moreover, for each $x \in M$, we have $\|T(x)\| \le a\|x\| + b + c$, and hence

$$\|T^n(x)\| \le a^n \|x\| + \left(\sum_{i=0}^{n-1} a^i\right)(b+c)\ \forall n \ge 1.$$

It then follows that $B = \{x \in M : \|x\| \le \frac{b+c}{1-a}\}$ is positively invariant for T, and attracts every bounded subset of M for T. So $T : (M,d) \to (M,d)$ has a strong global attractor.

Let $\varepsilon > 0$ be fixed such that $a + \frac{c}{1+\varepsilon} > 1$. We claim that

$$\limsup_{n\to\infty} \|T^n x\| \ge \varepsilon\ \forall x = (\varphi, y) \in M_0.$$

Assume, by contradiction, that $\limsup_{n\to\infty} \|T^n x\| < \varepsilon$ for some $x = (\varphi, y) \in M_0$. We set $(\varphi_n, y_n) = T^n x\ \forall n \ge 0$. By the definition of M, we have $\varphi \in L^1_+((0,+\infty), \mathbb{R})\setminus\{0\}$. It then follows that there exists $n_0 \ge 0$ such that $\int_0^1 \varphi_{n_0}(l)dl > 0$ and

$$\int_0^1 \varphi_{n+1}(l)dl \ge \left(a + \frac{c}{1+\varepsilon}\right)\int_0^1 \varphi_n(l)dl\ \forall n \ge n_0.$$

Thus, we obtain

$$\int_0^1 \varphi_n(l)dl \to +\infty \text{ as } n \to +\infty,$$

which is a contradiction. By Proposition 3.2, we conclude that T is ρ-uniformly persistent. Since $T : (M,d) \to (M,d)$ has a global attractor, it follows from Theorem 3.7 that $T : (M_0, d_0) \to (M_0, d_0)$ has a global attractor.

To avoid possible confusion, we denote by κ_0 the Kuratowski measure of non-compactness on the complete metric (M_0, d_0). We now consider $T : (M_0, d_0) \to (M_0, d_0)$. Let $\varepsilon > 0$ be fixed such that

$$\sqrt{a} < d := a + \frac{b}{1+\varepsilon} < 1.$$

Then for each $x \in M$, we have

$$\|T(x)\| \ge a\|x\| + b\frac{\|x\|}{1+\|x\|} \ge d\min(\varepsilon, \|x\|).$$

Let $B \subset M_0$ be a ρ-bounded set. We set $\rho_0 = \inf_{x\in B} \rho(x)$. Then for each $x \in B$, we obtain

$$\|T(x)\| \ge a\|x\| + b\frac{\|x\|}{1+\|x\|} \ge d\min(\varepsilon, \|x\|).$$

By induction, it follows that

$$\rho(T^n(x)) \ge d^n \min(\varepsilon, \rho_0)\ \ \forall n \ge 1, \forall x \in B.$$

Thus, for each $x, y \in B$, we have

$$\begin{aligned} d_0\left(T^n(x), T^n(y)\right) &= \left|\frac{1}{\rho\left(T^n\left(x\right)\right)} - \frac{1}{\rho\left(T^n\left(y\right)\right)}\right| + \left\|T^n\left(x\right) - T^n\left(y\right)\right\| \\ &\leq \left[\frac{1}{\rho\left(T^n\left(x\right)\right)\rho\left(T^n\left(y\right)\right)} + 1\right]\left\|T^n\left(x\right) - T^n\left(y\right)\right\| \\ &\leq \left[\frac{1}{d^{2n}\min\left(\varepsilon, \rho_0\right)^2} + 1\right] d(T^n\left(x\right), T^n\left(y\right)), \end{aligned}$$

and hence,

$$\begin{aligned} \kappa_0\left(T^n(B)\right) &\leq \left[\frac{1}{d^{2n}\min\left(\varepsilon, \rho_0\right)^2} + 1\right]\kappa\left(T^n(B)\right) \\ &\leq a^n\left[\frac{1}{d^{2n}\min\left(\varepsilon, \rho_0\right)^2} + 1\right]\kappa\left(B\right). \end{aligned}$$

Since $d > \sqrt{a}$, we obtain $\kappa_0\left(T^n(B)\right) \to 0$ as $n \to +\infty$. So $T : (M_0, d_0) \to (M_0, d_0)$ is κ_0-contracting.

It remains to show that $T : (M_0, d_0) \to (M_0, d_0)$ has no strong global attractor. Let $\delta > 0$ be fixed, and consider the ρ-strongly bounded set

$$B_\delta = \left\{x \in M : \rho\left(x\right) = \delta\right\}.$$

For each $m \geq 0$, we set $x^m := (\varphi^m, 0)$ with $\varphi^m = \delta 1_{[m,m+1]}$, and

$$x_n^m := (\varphi_n^m, y_n^m) = T^n\left(x^m\right) \quad \forall n \geq 0.$$

Then for each $m \geq 1$ and each $n \in \{0, \ldots, m-1\}$, we have $\int_0^1 \varphi_n^m(l)dl = 0$, and hence,

$$\begin{cases} \varphi_{n+1}^m(\cdot) = a\varphi_n^m(\cdot + 1) + a\int_0^1 \varphi_n^m(l)dl 1_{[0,1]}(\cdot), \\ y_{n+1}^m = a y_n^m + b\frac{\|x_n^m\|}{1+\|x_n^m\|}. \end{cases}$$

Thus, for each $m \geq 1$ and each $n \in \{0, \ldots, m-1\}$, we obtain

$$\left\|x_{n+1}^m\right\| \leq (a+b)\left\|x_n^m\right\| \leq (a+b)^n \delta.$$

It follows that $\inf_{x \in B_\delta} \rho\left(T^n\left(x\right)\right) \to 0$ as $n \to +\infty$. So the κ_0-contracting map $T : (M_0, d_0) \to (M_0, d_0)$ has a global attractor, but no strong global attractor.

5.3. $\boldsymbol{\kappa}$-contracting semiflows on $\boldsymbol{(M_0, d_0)}$. In this subsection, we construct continuous-time κ-contracting semiflows on (M_0, d_0) such that they admit a global attractor, but no strong global attractor.

Let X and X_+ be defined as in the previous subsection. Consider the following age-structured model:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -\mu u(t, a),\ t \geq 0, a \in (0, \infty), \\ u(t, 0) = \frac{\int_0^{+\infty} \beta(a) u(t,a) da}{1+\|(u(t), y(t))\|}, \\ \frac{dy(t)}{dt} = -\mu y(t) + \gamma\frac{\|(u(t),y(t))\|}{1+\|(u(t),y(t))\|}, \\ u(0, .) = u_0 \in L_+^1\left((0, +\infty), \mathbb{R}\right),\ y(0) = y_0 \in \mathbb{R}_+. \end{cases} \tag{4}$$

We assume that

(A4) $\mu > 0$, $\gamma \in (\frac{\mu}{2}, \mu), \beta : \mathbb{R}_+ \to \mathbb{R}_+$ is uniformly continuous, bounded, $\int_0^\infty \beta(a)e^{-\mu a}da > 1$, and there exists a sequence of real numbers $\{a_n\}_{n\geq 0} \subset [0, +\infty)$ such that $a_n < a_{n+1}$ $\forall n \geq 0$, $\lim_{n\to+\infty}(a_{2n+2} - a_{2n+1}) = +\infty$, and

$$\beta(a) > 0 \Leftrightarrow a \in \bigcup_{n\geq 0} (a_{2n}, a_{2n+1}).$$

For each $\chi \in L^\infty((0, +\infty), \mathbb{R})$ and each $\varphi \in L^1((0, +\infty), \mathbb{R})$, we define

$$\mathcal{F}_\chi(\varphi) = \int_0^{+\infty} \chi(s)\varphi(s)ds.$$

Let $\{U(t)\}_{t\geq 0}$ be the solution semiflow on X_+ generated by system (4), and let $(u(t), y(t)) = U(t)(u_0, y_0)$. Then we have the following Volterra formulation of system (4):

$$u(t, a) = \begin{cases} e^{-\mu t}u_0(a - t) \text{ if } a > t, \\ e^{-\mu a}B(t - a) \text{ if } a \leq t, \end{cases}$$

with $B(t) = \frac{\mathcal{F}_\beta(u(t))}{1+\mathcal{F}_1(u(t))+y(t)}$, and for each $t \geq 0$,

$$\begin{cases} \frac{d\mathcal{F}_1(u(t))}{dt} = -\mu\mathcal{F}_1(u(t)) + \frac{\mathcal{F}_\beta(u(t))}{1+\mathcal{F}_1(u(t))+y(t)}, \\ \frac{dy(t)}{dt} = -\mu y(t) + \gamma\frac{\mathcal{F}_1(u(t))+y(t)}{1+\mathcal{F}_1(u(t))+y(t)}, \end{cases} \tag{5}$$

and

$$\mathcal{F}_\beta(u(t)) = e^{-\mu t}\int_t^{+\infty} \beta(s)u_0(s - t)ds + \int_0^t \beta(s)e^{-\mu a}\frac{\mathcal{F}_\beta(u(t - a))}{1 + \mathcal{F}_1(u(t - a)) + y(t - a)}da.$$

LEMMA 5.2. *There exists a continuous and nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, $f(x) > 0$ $\forall x > 0$, and the set*

$$M := \{(\varphi, y) \in X_+ : y \leq f(\|\varphi\|)\}$$

is positively invariant for $\{U(t)\}_{t\geq 0}$.

Proof. Let $(\widehat{x}(t), \widehat{y}(t))$ be the unique solution on $[0, \infty)$ of the following cooperative system:

$$\begin{cases} \frac{dx(t)}{dt} = -\mu x(t), \\ \frac{dy(t)}{dt} = -\mu y(t) + \gamma\frac{x(t)+y(t)}{1+x(t)+y(t)} \end{cases} \tag{6}$$

with

$$(\widehat{x}(0), \widehat{y}(0)) = \left(\frac{\gamma}{\mu} + 1, \frac{\gamma}{\mu} + 1\right).$$

Since $\widehat{x}'(0) < 0$ and $\widehat{y}'(0) < 0$, $(\widehat{x}(t), \widehat{y}(t))$ is nonincreasing on some small interval $[0, \epsilon]$. By the monotonicity of the solution semiflow of system (6) on $\mathbb{R}^2_+$, it follows that $(\widehat{x}(t), \widehat{y}(t))$ is nonincreasing on $[0, \infty)$, and $(\widehat{x}(t), \widehat{y}(t)) \to (0, 0)$ as $t \to +\infty$. Set

$$\widehat{x}(t) = \frac{\gamma}{\mu} + 1 - t, \text{ and } \widehat{y}(t) = \frac{\gamma}{\mu} + 1 \quad \forall t \in (-\infty, 0].$$

Clearly, $\widehat{x}(t)$ is strictly decreasing in $t \in \mathbb{R}$. Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f(\alpha) = \begin{cases} \widehat{y}(\widehat{x}^{-1}(\alpha)) \text{ if } \alpha > 0, \\ 0 \text{ if } \alpha = 0. \end{cases}$$

Then f satisfies the desired properties. Note that the set $D := \{(x, y) \in \mathbb{R}^2_+ : y \leq f(x)\}$ is positively invariant for the solution semiflow of (6). By using the monotonicity of f and the planar vector field associated with (5), one can easily prove that $U(t)M \subset M$ $\forall t \geq 0$. $\square$

Now we consider $U(t) : (M, d) \to (M, d)$, where $d(x, \widehat{x}) = \|x - \widehat{x}\|$. Set

$$\partial M_0 = \{0\}, \ M_0 = M \setminus \{0\}, \text{ and } \rho(x) = \|x\|.$$

Since $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is uniformly continuous, it follows from [28] that for any bounded set $B \subset M$, we have

$$\kappa(U(t)B) \leq e^{-\mu t}\kappa(B) \ \forall t \geq 0.$$

Let $z(t) := \mathcal{F}_1(u(t)) + y(t)$. Then we obtain

$$\frac{dz(t)}{dt} \leq -\mu z(t) + (\|\beta\|_\infty + \gamma) \ \forall t \geq 0.$$

Consequently, $U(t) : (M, d) \to (M, d)$ has a strong global attractor.

Let $\varepsilon > 0$ be such that

$$\frac{\int_0^{+\infty} \beta(a)e^{-\mu a}da}{1 + \varepsilon} > 1.$$

We claim that $\limsup_{t\to\infty} \|U(t)x\| \geq \varepsilon$ $\forall x \in M_0$. Assume, by contradiction, that $\limsup_{t\to\infty} \|U(t)x\| < \varepsilon$ for some $x = (u_0, y_0) \in M_0$. Then there exists $t_0 \geq 0$ such that $\|U(t + t_0)x\| < \varepsilon$ $\forall t \geq 0$. By the definition of M, we have $u_0 \neq 0$, and hence, $u(t) \neq 0$ $\forall t \geq 0$. It follows that $u(t + t_0) \geq \widehat{T}(t)u(t_0)$ $\forall t \geq 0$, where $\{\widehat{T}(t)\}_{t\geq 0}$ is the strongly continuous semigroup of bounded linear operators on $L^1((0, +\infty), \mathbb{R})$, which is generated by $\widehat{A}\varphi = -\varphi' - \mu\varphi$ with

$$D(\widehat{A}) = \left\{ \varphi \in W^{1,1}((0, +\infty), \mathbb{R}) : \varphi(0) = \frac{\int_0^{+\infty} \beta(a)\varphi(a)da}{1 + \varepsilon} \right\}.$$

Since $u(t_0) \neq 0$, it follows from [28] that

$$\|u(t + t_0)\|_{L^1} \geq \left\|\widehat{T}(t)u(t_0)\right\|_{L^1} \to +\infty \text{ as } t \to +\infty,$$

which is a contradiction. By the continuous-time version of Proposition 3.2, we deduce that $U(t) : (M, d) \to (M, d)$ is ρ-uniformly persistent, and hence, $U(t) : (M_0, d_0) \to (M_0, d_0)$ has a global attractor (see Theorem 3.7 and Remark 3.10).

We now prove that $U(t) : (M_0, d_0) \to (M_0, d_0)$ is κ_0-contracting. Let $\varepsilon > 0$ be such that $\mu - \frac{2\gamma}{1+\varepsilon} < 0$. Let B be a ρ-strongly bounded set of M_0. We set $\rho_0 = \inf_{x\in B} \rho(x)$. For each $x \in B$, if we set $z(t) = \rho(U(t)x)$ $\forall t \geq 0$, we then have

$$\frac{dz(t)}{dt} \geq -\mu z(t) + \gamma\frac{z(t)}{1 + z(t)} \ \forall t \geq 0,$$

and hence,

$$\rho\left(U(t)x\right) \geq e^{\left(-\mu+\frac{\gamma}{1+\varepsilon}\right)t}\min\left(\varepsilon,\rho_0\right)\ \forall t\geq 0.$$

It then follows that for each $x, y \in B$, we have

$$d_0\left(U(t)x,U(t)y\right) \leq \left[\frac{1}{e^{2\left(-\mu+\frac{\gamma}{1+\varepsilon}\right)t}\min\left(\varepsilon,\rho_0\right)^2}+1\right]d\left(U(t)x,U(t)y\right),$$

and hence,

$$\begin{aligned}\kappa_0\left(U(t)B\right) &\leq \left[\frac{1}{e^{2\left(-\mu+\frac{\gamma}{1+\varepsilon}\right)t}\min\left(\varepsilon,\rho_0\right)^2}+1\right]\kappa\left(U(t)B\right)\\ &\leq e^{-\mu t}\left[\frac{1}{e^{2\left(-\mu+\frac{\gamma}{1+\varepsilon}\right)t}\min\left(\varepsilon,\rho_0\right)^2}+1\right]\kappa\left(B\right).\end{aligned}$$

Since $\mu-\frac{2\gamma}{1+\varepsilon}<0$, we deduce that $\kappa_0\left(U(t)B\right)\to 0$ as $t\to+\infty$. So $U(t):(M_0,d_0)\to(M_0,d_0)$ is κ_0-contracting.

It remains to show that $U(t):(M_0,d_0)\to(M_0,d_0)$ has no strong global attractor. We fix a real number $\delta>0$ and set

$$B := \left\{x\in M:\rho\left(x\right)=\delta\right\}.$$

Let $x^n=(u_0^n,0)$, with $u_0^n=\delta 1_{[a_{2n+1},a_{2n+1}+1]}(\cdot)$, and $(u^n(t),y^n(t))=U(t)x^n\ \forall t\geq 0$. Then for each $t\geq 0$, we have

$$\begin{aligned}\mathcal{F}_\beta\left(u^n(t)\right) &= e^{-\mu t}\int_t^{+\infty}\beta(s)u_0^n(s-t)ds\\ &+\int_0^t\beta(s)e^{-\mu a}\frac{\mathcal{F}_\beta\left(u^n(t-a)\right)}{1+\mathcal{F}_1\left(u^n(t-a)\right)+y^n(t-a)}da\end{aligned}\tag{7}$$

and

$$\begin{aligned}\int_t^{+\infty}\beta(s)u_0^n(s-t)ds &= \int_0^{+\infty}\beta(s+t)u_0^n(s)ds\\ &=\delta\int_{a_{2n+1}}^{a_{2n+1}+1}\beta(s+t)ds=\delta\int_{t+a_{2n+1}}^{t+a_{2n+1}+1}\beta(s)ds.\end{aligned}$$

Since $a_{2n+2}-a_{2n+1}\to+\infty$ as $n\to+\infty$, there exists $n_0\geq 0$ such that $a_{2n+2}-a_{2n+1}>1\ \forall n\geq n_0$. Then we have

$$\int_t^{+\infty}\beta(s)u_0^n(s-t)ds=0\ \forall t\in[0,a_{2n+2}-(a_{2n+1}+1)],\ \forall n\geq n_0.$$

Since $\mathcal{F}_\beta\left(u^n(t)\right)$ is a solution of (7), we deduce that for each $n\geq n_0$, and $t\in[0,a_{2n+2}-(a_{2n+1}+1)]$, $\mathcal{F}_\beta\left(u^n(t)\right)=0$. It then follows that $z_n(t):=\|U(t)x^n\|$ satisfies $z_n(0)=\delta$ and

$$\frac{dz_n(t)}{dt}=-\mu z_n(t)+\gamma\frac{z_n(t)}{1+z_n(t)}\ \forall t\in\left[0,a_{2n+2}-(a_{2n+1}+1)\right],\ \forall n\geq n_0.$$

Thus, we have

$$z_n(t)\leq e^{(-\mu+\gamma)t}\delta\ \forall t\in\left[0,a_{2n+2}-(a_{2n+1}+1)\right],$$

which implies that $\inf_{x\in B}\rho\left(U(t)x\right)\to 0$, as $t\to+\infty$. So $U(t):(M_0,d_0)\to(M_0,d_0)$ has no strong global attractor.

5.4. A periodic age-structured model. In this subsection, we illustrate applicability of Theorem 4.5 in the case of convex κ-contracting maps.

Consider the 1-periodic nonautonomous age-structured model

$$(8)\quad \begin{cases} \frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}=-\left(\mu+m(t,\int_0^{+\infty}u(t,l)dl)(a)\right)u(t,a),\ t\geq 0,\ a\in(0,+\infty)\,, \\ u(t,0)=\frac{\int_0^{+\infty}\beta(t,a)u(t,a)da}{1+\int_0^{+\infty}u(t,a)da}, \\ u(0,.)=u_0\in L^1_+\left((0,+\infty),\mathbb{R}\right). \end{cases}$$

We assume that

(A5) $\mu>0$ and the following conditions are satisfied:

(a) $\beta:\mathbb{R}^2_+\to\mathbb{R}_+$ is uniformly continuous, positive, bounded, and $t\to\beta(t,a)$ is 1-periodic.

(b) $m\in C\left(\mathbb{R}^2_+,L^\infty_+\left((0,+\infty),\mathbb{R}\right)\right)$ and the map $t\to m(t,\cdot)$ is 1-periodic.

(c) There exist a bounded and uniformly continuous map $\widehat{\beta}:\mathbb{R}_+\to\mathbb{R}_+$ and a continuous and bounded map $\widehat{m}:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\beta(t,\cdot)\geq\widehat{\beta}(\cdot)\text{ and }m(t,\cdot)\leq\widehat{m}(\cdot)\ \forall t\in[0,1]\,,$$

and for any $a\geq 0$, there exists $r\geq a$ such that $\widehat{\beta}(r)>0$ and

$$\int_0^{+\infty}\widehat{\beta}(a)e^{-\int_0^a\mu+\widehat{m}(r)dr}da>1.$$

Let $Y=L^1_+\left((0,+\infty),\mathbb{R}\right)$ and $Y_+=L^1_+\left((0,+\infty),\mathbb{R}\right)$, and let $\{U(t,s)\}_{0\leq s\leq t}$ be the nonautonomous semiflow generated by system (8). Set

$$M=Y_+,\ \partial M_0=\{0\}\ \text{ and }M_0=Y_+\setminus\{0\}\,.$$

Then $U(t,s)0=0$, and $U(t,s)M_0\subset M_0\ \forall t\geq s\geq 0$. For a 1-periodic solution of system (8) in $Y_+\setminus\{0\}$, it suffices to find a fixed point of $T=U(1,0)$. By setting $x(t):=\mathcal{F}_1\left(U(t,s)x\right)$, we have

$$\frac{dx(t)}{dt}\leq-\mu x(t)+\|\beta\|_\infty\frac{x(t)}{1+x(t)},$$

which implies that T is bounded dissipative on M. Moreover, by using the results in [28] and assumptions (A5)(a),(b), we obtain

$$U(t,s)=C(t,s)+N(t,s),$$

where $C(t,s)$ is a compact operator, and

$$\|N(t,s)x\|\leq e^{-\mu(t-s)}\|x\|\ \ \forall t\geq s\geq 0,\forall x\in M.$$

Thus, T is κ-contracting in the sense that $\kappa\left(T^n\left(B\right)\right)\to 0$ as $n\to+\infty$ for any bounded set $B\subset M$. It follows from Theorem 2.9 that T has a strong global attractor in M. Using assumption (A5)(c) and comparison arguments, we can further prove that the fixed point 0 of T is ejective. In order to apply Theorem 4.5, we need to verify that T is convex κ-contracting.

Let $V(t,s) = (V_1(t,s), V_2(t,s))$ be the nonautonomous semiflow on $Y_+ \times Y_+$, which is generated by the following system:

$$\begin{cases} \frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial a} = -\left(\mu + m(t, \int_0^{+\infty} (u_1 + u_2)(t,l)dl)(a)\right) u_1(t,a),\ a \in (0,+\infty), \\ u_1(t,0) = 0, \\ \frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial a} = -\left(\mu + m(t, \int_0^{+\infty} (u_1 + u_2)(t,l)dl)(a)\right) u_2(t,a),\ a \in (0,+\infty), \\ u_2(t,0) = \frac{\int_0^{+\infty} \beta(t,a)(u_1+u_2)(t,a)da}{1+\alpha\int_0^{+\infty}(u_1+u_2)(t,a)da}, \\ (u_1(0,.), u_2(0,.)) = \left(u_0^1, u_0^2\right) \in L_+^1\left((0,+\infty), \mathbb{R}^2\right). \end{cases}$$

We define $P_n : Y \to Y$ by

$$P_n(\varphi) = \varphi 1_{[n,+\infty)}\ \forall n \geq 0.$$

Then for each $n \geq 0$, we have

$$P_{n+1}T(x) = V_1(1,0)(P_n x, (I - P_n)x)$$

and

$$(I - P_{n+1})T(x) = V_2(1,0)(P_n x, (I - P_n)x).$$

Moreover, if B is bounded and $(I - P_n)(B)$ is relatively compact, then

$$\{(I - P_{n+1})T(x) :\ x \in B\} = \{V_2(1,0)(P_n x, (I - P_n)x) :\ x \in B\}$$

is relatively compact. Note that for each $x \in M$, we have

$$\|P_{n+1}T(x)\| = \|V_1(1,0)(P_n x, (I - P_n)x)\| \leq e^{-\mu}\|P_n x\|.$$

By Lemma 4.8, it follows that T is convex κ-contracting. Thus, Theorem 4.5 implies that T has a fixed point in M_0, and hence, system (8) admits a nontrivial 1-periodic solution.

Finally, we remark that the similar approach can be applied to more general age-structured models.

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