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THE QUALITATIVE ANALYSIS OF N-SPECIES LOTKA-VOLTERRA PERIODIC COMPETITION SYSTEMS

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Abstract—In this paper, we consider an n-species Lotka-Volterra periodic competition system. Using a comparison method and the Brouwer fixed point theorem, we obtain some sufficient conditions for the ultimate boundedness of solutions and the existence and global attractivity of a positive periodic solution. We also point out that these results constitute a generalization of K. Gopalsamy and J. M. Cushing's.

1. INTRODUCTION

Most mathematical models for the dynamics of population growth are autonomous, which is to say that they attempt to describe the growth and interaction of species with constant vital parameters living in a constant environment. While this hypothesis of constant environment and vital parameters is justifiable under some circumstances, a more realistic model would certainly allow for the temporal variation of these parameters. Much of this temporal variation could naturally be assumed to be periodic due to seasonal (or other periodic) effects of food availability, weather conditions, temperature, mating habits, etc.

In this paper, we consider the n-species Lotka-Volterra system

$$\frac{dx_i(t)}{dt} = x_i(t)(b_i(t) - \sum_{J=1}^{n} a_{ij}(t)x_j(t)), \quad n \geq 1,\, x_i \geq 0, \quad i = 1, 2 \ldots, n, \tag{1}$$

where $b_i(t)$, $a_{ij}(t)\,(i, j = 1, \ldots, n)$ are continuous ω-periodic functions with $\int_o^\omega b_i(t)\,dt > 0$ and $a_{ij}(t) > 0$. In Section 2, it is proved that any solution of (1) with positive initial value is ultimately bounded. In Section 3, some sufficient conditions for the existence and global attractivity of positive periodic solution of (1) are obtained.

2. ULTIMATE BOUNDEDNESS OF SOLUTIONS

Consider the Logistic equation

$$\frac{dx(t)}{dt} = x(t)(b(t) - a(t)x(t)), \qquad x \in \mathbf{R}, \tag{2}$$

where $a(t)$ and $b(t)$ are continuous ω-periodic functions. From the change of variable $y = 1/x$ and explicit solution of the resulting equation, we can prove the following

LEMMA. *If $\int_0^\omega b(t)\,dt > 0, a(t) > 0$, then Equation (2) has a unique globally attracting positive stable ω-periodic solution. Moreover, let $x_1(t)$ and $x_2(t)$ be the unique positive solution of (2) with $b(t) = b_1(t), b_2(t)$ respectively. If $b_1(t) > b_2(t)$, then $x_1(0) > x_2(0)$.*

If $g(t)$ is a continuous ω-periodic function, we denote $\max\limits_{0 \leq t \leq \omega} g(t)$ by g^u and $\min\limits_{0 \leq t \leq \omega} g(t)$ by g^l. Choose $K_i > 0$ such that $K_i > b_i^u/a_{ii}^l\,(i = 1, 2, \ldots, n)$. Denote

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$$S = \{(x_1, x_2, \dots, x_n); 0 < x_1 < K_i, (i = 1, 2, \dots, n)\}.$$

THEOREM 1. *Any solution $(x_1(t), \dots, x_n(t))$ with $x_i(0) >$ $(i = 1, 2, \dots, n)$ of (1) ultimately lies in S.*

PROOF. Let $\overline{x}_i(t)$ be the unique positive periodic solution of the Logistic equation

$$\frac{dx_i}{dt} = x_i(b_i(t) - a_{ii}(t)\, x_i(t)), \qquad (i = 1, 2, \dots, n). \tag{3}$$

If $\overline{x}_i(t)$ attains its maximum when $t = t_i$, then $\frac{d\overline{x}_i(t_i)}{dt} = 0$, and hence

$$\overline{x}_i(t_i) = \frac{b_i(t_i)}{a_{ii}(t_i)} \le \frac{b_i^u}{a_{ii}^l}.$$

Then $\overline{x}_i(t) \le b_i^u / a_{ii}^l$. Let $u_i(t)$ be the solution of (3) with $u_i(0) = x_i(0)$, by Lemma,

$$\lim_{t \to +\infty} (u_i(t) - \overline{x}_i(t)) = 0.$$

As

$$\frac{dx_i(t)}{dt} = x_i(t)(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\, x_j(t)) \le x_i(t)(b_i(t) - a_{ii}(t))$$

by the differential inequality theorem [1],

$$0 < x_i(t) \le u_i(t), \qquad 0 \le t \le +\infty.$$

Take $\epsilon_i = K_i - b_i^u / a_{ii}^l > 0$, then there exists $T = T(x_1(0), \dots, x_n(0))$ such that when $t \ge T$,

$$|u_i(t) - \overline{x}_i(t)| < \epsilon_i, \qquad (i = 1, 2, \dots n).$$

Then

$$u_i(t) = u_i(t) - \overline{x}_i(t) + \overline{x}_i(t) < \epsilon_i + \frac{b_i^u}{a_{ii}^l} = K_i.$$

Therefore, when $t \ge T$, $0 < x_i(t) < K_i$ $(i = 1, 2, \dots, n)$, i.e., $(x_1(t), x_2(t), \dots, x_n(t)) \in S$. This completes our proof.

From the process of the proof above, it follows that any solution $x(t) = (x_i(t), \dots, x_n(t))$ with $x_i(0) > 0 (i = 1, 2, \dots, n)$ of (1) is defined on $[0, +\infty)$.

3. THE EXISTENCE AND GLOBAL ATTRACTIVITY OF THE PERIODIC SOLUTION

THEOREM 2. *Let $\overline{x}_i(t)$ be the unique positive periodic solution of (3) $(i = 1, 2, \dots, n)$. If*

$$\int_0^\omega b_i(t)\, dt > \sum_{j \ne i, j=1}^{n} \int_0^\omega a_{ij}(t)\, \overline{x}_j(t)\, dt, \qquad (i = 1, 2, \dots n),$$

then Equation (1) has at least one positive (componentwise) periodic solution.

PROOF. Let

$$B_i(t) = b_i(t) - \sum_{j \ne i, j=1}^{n} a_{ij}(t)\, \overline{x}_j(t), \qquad (i = 1, 2, \dots, n),$$

then $B_i(t + \omega) = B_i(t)$ and $\int_0^\omega B_i(t)\, dt > 0$. Let $\tilde{x}_i(t)$ be the unique positive periodic solution of the Logistic equation

$$\frac{dx_i}{dt} = x_i(B_i(t) - a_{ii}(t)\, x_i), \qquad (i = 1, 2, \dots, n). \tag{4}$$

As $b_i(t) > B_i(t)$, by Lemma, then $\overline{x}_i(0) > \tilde{x}_i(0)$. Therefore the set

$$D = \{x_1, \dots, x_n);\ \tilde{x}_i(0) \le x_i \le \overline{x}_i(0), \quad (i = 1, 2, \dots, n)\}$$

makes sense. Define Poincaré mapping $A : R^n \longrightarrow R^n$ as follows

$$A(x^0) = x(\omega, x^0), \qquad x^0 \in R^n,$$

where $x(t, x^0)$ is the solution with $x(0, x^0) = x^0$ of (1). Because (1) is ω-periodic system, it follows that if $x^* = A(x^*)$, then $x(t, x^*)$ is the ω-periodic solution of (1). For every $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$, let $x_i(t) = x_i(t, x^0)(i = 1, 2, \dots, n)$, then

$$\frac{dx_i(t)}{dt} = x_i(t)(b_i(t) - \sum_{j=1}^{n} a_{ij}(t)\, x_j(t)) \le x_i(t)(b_i(t) - a_{ii}(t)\, x_i(t)).$$

As $x_i(0) = x_i^0 \le \overline{x}_i(0)$, by differential inequality theorem

$$x_i(t) \le \overline{x}_i(t), \quad 0 \le t < +\infty.$$

Therefore

$$\frac{dx_i(t)}{dt} \ge x_i(t)(b_i(t) - \sum_{j \ne i, j=1}^{n} a_{ij}(t)\, \overline{x}_j(t) - a_{ii}(t)\, x_i(t)) = x_i(t)(B_i(t) - a_{ii}(t)\, x_i(t)).$$

As $x_i(0) = x_i^0 \ge \tilde{x}_i(0)$, then

$$x_i(t) \ge \tilde{x}_i(t), \quad 0 \le t < +\infty.$$

Therefore

$$\tilde{x}_i(t) \le x_i(t) \le \overline{x}_i(t), \quad 0 \le t < +\infty, \qquad (i = 1, 2, \dots, n).$$

Then

$$\tilde{x}_i(0) = \tilde{x}_i(\omega) \le x_i(\omega) \le \bar{x}_i(\omega) = \overline{x}_i(0), \qquad (i = 1, 2, \dots\ n),$$

therefore $A(x^0) \in D$, i.e., $AD \subset D$. By Brouwer's fixed point theorem, there exists $x^* \in D$ such that $Ax^* = x^*$. Therefore, $x(t, x^*)$ is the positive ω-periodic solution of (1). This completes our proof.

COROLLARY 1. *Suppose that* $\int_0^\omega b_i(t)\, dt > 0, a_{ij}$ *are positive constants and*

$$\int_0^\omega b_i(t)\, dt > \sum_{j \ne i, j=1}^{n} \frac{a_{ij}}{a_{jj}} \int_0^\omega b_j(t)\, dt, \qquad (i = 1, 2, \dots, n),$$

then Equation (1) has at least one positive periodic solution.

PROOF. Because $\overline{x}_i(t) > 0$,

$$\frac{\overline{x}_i(t)}{dt} = \overline{x}_i(t)(b_i(t) - a_{ii}(t)\, \overline{x}_i(t)),$$

then

$$\frac{1}{\overline{x}_i(t)} \frac{dx_i(t)}{dt} = b_i(t) - a_{ii}\, \overline{x}_i(t).$$

Integrate both sides of this equality over $[0, \omega]$, we have

$$\int_0^\omega b_i(t)\, dt = a_{ii} \int_0^\omega \overline{x}_i(t)\, dt.$$

Then

$$\int_0^\omega \overline{x}_i(t)\,dt = \frac{\int_0^\omega b_i(t)dt}{a_{ii}}, \qquad (i = 1, 2, \dots, n).$$

Therefore

$$\int_0^\omega b_i(t)\,dt > \sum_{j\neq i,\,j=1}^{n} \frac{a_{ij}}{a_{jj}} \int_0^\omega b_j(t)\,dt = \sum_{j\neq i,\,j=1}^{n} \int_0^\omega a_{ij}\,\overline{x}_j(t)\,dt.$$

By Theorem 2, Corollary 1 holds.

COROLLARY 2. *(K. Gopalsamy [2]). Suppose that $b_i(t) > 0$, $a_{ij}(t) > 0$ and*

$$b_i^l > \sum_{j\neq i,\,j=1}^{n} a_{ij}^u \frac{(b_j^u)}{(a_{jj}^l)}, \qquad (i = 1, 2, \dots, n),$$

then (1) has at least one positive periodic solution.

PROOF. As $b_i(t) > 0$, then $\int_0^\omega b_i(t)\,dt > 0$. From the proof of Theorem 1, it follows that

$$\overline{x}_j(t) \le \frac{b_j^u}{a_{jj}^l}, \qquad (i = 1, 2, \dots, n).$$

Then

$$b_i(t) \ge b_i^l > \sum_{j\neq i,\,j=1}^{n} a_{ij}^u \frac{(b_j^u)}{(a_{jj}^l)} \ge \sum_{j\neq i,\,j=1}^{n} a_{ij}(t)\,\overline{x}_j(t).$$

Therefore

$$\int_0^\omega b_i(t)\,dt > \sum_{j\neq i,\,j=1}^{n} \int_0^\omega a_{ij}(t)\,\overline{x}_j(t)\,dt, \qquad (i = 1, 2, \dots, n).$$

By Theorem 2, Corollary 2 holds.

NOTE 1. For Equation (1), with $n = 2$, J.M. Cushing ([3], Theorem 1), using bifurcation theory, proved that a branch of positive periodic solution exists for a finite interval of values of bifurcation parameter $[b_2] = \frac{1}{\omega}\int_0^\omega b_2(t)dt$, with $\mu_1 = [a_{21}\overline{x}_1(t)]$ as one of its endpoints. However, another endpoint and the bifurcation direction are not determined. As the conditions of Theorem 2 for $n = 2$ become

$$\int_0^\omega b_1(t)\,dt > \int_0^\omega a_{12}(t)\,\overline{x}_2(t)\,dt, \tag{C1}$$

$$\int_0^\omega b_2(t)\,dt > \int_0^\omega a_{21}(t)\,\overline{x}_1(t)\,dt. \tag{C2}$$

Therefore (C2) means $\mu > \mu_1$. Clearly, another endpoint relates to (C1). For example, if $a_{ij}(i = 1, 2)$ are positive constants, then $\mu_1 = (a_{21}/a_{11})[b_1]$, and (C1) means (from Corollary 1)

$$\mu < \mu_2 = \frac{a_{22}}{a_{12}}[b_1],$$

and hence

$$\mu_1 < \mu < \mu_2.$$

In the following, we say that a periodic solution, say $u(t) = (u_1(t), \dots, u_n(t))$, of (1) is globally attractive if every other solution $x(t) = (x_1(t) \dots, x_n(t))$, with $x_i(t) > 0\,(i = 1, 2, \dots, n)$ of (1) is defined for all $t \ge 0$ and satisfies

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0, \qquad (i = 1, 2, \dots, n).$$

THEOREM 3. *Suppose that the conditions in Theorem 2 hold, and if, in addition,*

$$a_{jj}(t) > \sum_{i\neq j,\, i=1}^{n} a_{ij}(t), \quad t \in [0,\omega], \qquad (j = 1,2,\dots,n),$$

then Equation (1) has a unique positive periodic solution which is globally attractive.

PROOF. By Theorem 2, (1) has a positive periodic solution $u(t) = (u_1(t),\dots,u_n(t))$. Let $x(t) = (x_1(t), x_2(t),\dots,x_n(t))$ be any other solution with $x_i(0) > 0\,(i = 1,2\dots,n)$ of (1). By the additional condition given, since $a_{ij}(t)$ are continuous and ω-periodic functions, we can choose $\alpha > 0$ such that

$$a_{jj}(t) - \sum_{i\neq j, i=1}^{n} a_{ij}(t) \geq \alpha, \qquad t \in [0,+\infty), \qquad (j = 1,2,\dots,n).$$

Let $X_i(t) = \log x_i(t), U_i(t) = log\, u_i(t)$. Consider continuous function

$$V(t) = \sum_{i=1}^{n} |X_i(t) - U_i(t)|.$$

As $u(t)$ and $x(t)$ are two solutions of (1), then

$$\frac{d}{dt}(X_i(t) - U_i(t)) = -a_{ii}(t)(x_i(t) - u_i(t)) - \sum_{j\neq i, j=1}^{n} a_{ij}(t)(x_j(t) - u_j(t)), \qquad (i = 1,2,\dots,n).$$

If $y(t)$ is any continuously differentiable scalar function defined on $[o,+\infty)$, we define a function σ by

$$\sigma_{(y)}(t) = \begin{cases} 1, & \text{if } y(t) > 0 \text{ or } y(t) = 0 \text{ and } \frac{dy(t)}{dt} > 0 \\ 0, & \text{if } y(t) = 0 \text{ and } \frac{dy(t)}{dt} = 0 \\ -1, & \text{if } y(t) < 0 \text{ or } y(t) = 0 \text{ and } \frac{dy(t)}{dt} < 0. \end{cases}$$

One can verify that

$$|y(t)| = y(t)\sigma_{(y)}(t),$$

and

$$D^+ |y(t)| = \sigma_{(y)}(t)\, \frac{dy(t)}{dt},$$

where D^+ denotes the upper right derivative. It follows from the above that

$$\begin{aligned} D^+V(t) &= \sum_{i=1}^{n} D^+|X_i(t) - U_i(t)| \\ &\leq \sum_{i=1}^{n} \left(-a_{ii}(t)|x_i(t) - u_i(t)| + \sum_{j\neq i, j=1}^{n} a_{ij}(t)|x_j(t) - u_j(t)| \right) \\ &= -\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{a}_{ij}(t)|x_j(t) - u_j(t)| \qquad (\overline{a}_{ij} = -a_{ij},\, i \neq j;\, \overline{a}_{ii} = a_{ii}) \\ &= -\sum_{j=1}^{n} \left(a_{jj}(t) - \sum_{i=1, i\neq j}^{n} a_{ij}(t) \right) |x_j(t) - u_j(t)| \\ &\leq -\alpha \sum_{j=l}^{n} |x_j(t) - u_j(t)| < 0. \end{aligned}$$

Then $V(t)$ is decreasing on $[0,+\infty)$. Therefore, $0 \le V(t) \le V(0)$ and $lim_{t\to+\infty} V(t) = V^* \ge 0$. Since $u(t)$ is a positive periodic function, then there exist two positive real numbers α_1 and α_2, such that

$$0 < \alpha_1 \le u_i(t) \le \alpha_2, \qquad (i = 1, 2, \ldots, n).$$

Then $U_i(t) = \log u_i(t)$ is bounded. As

$$|X_i(t)| \le |X_i(t) - U_i(t)| + |U_i(t)| \le V(t) + |U_i(t)|,$$

then $X_i(t)$ is also bounded. So we can assume that for some $M_0 > 0$,

$$|X_i(t)| \le M_0 \quad \text{and } |U_i(t)| \le M_0, \qquad (i = 1, 2, \ldots, n).$$

As

$$|x_i(t) - u_i(t)| = |e^{X_i(t)} - e^{U_i(t)}| = e^{\xi_i(t)}|X_i(t) - U_i(t)|,$$

then

$$m|X_i(t) - U_i(t)| \le |x_i(t) - u_i(t)| \le M|X_i(t) - U_i(t)|, \qquad (i = 1, 2, \ldots, n),$$

where $m = e^{-M_0}, M = e^{M_0}$. Then

$$D^+V(t) \le -\alpha m \sum_{i=1}^{n} |X_i(t) - U_i(t)| = -\alpha m V(t).$$

We claim that $V^* = 0$. Suppose $V^* > 0$, then $V(t) \ge V^* > 0$, for $t \in [0,+\infty)$. Therefore, $D^+V(t) \le -\alpha m V^*$. Then

$$V(t) \le V(0) - \alpha m V^* t \qquad t \in [0, +\infty).$$

If $t \ge V(0)/\alpha m V^*$, then $V(t) < 0$. This leads to a contradiction. Since

$$|x_i(t) - u_i(t)| \le M|X_i(t) - U_i(t)| \le MV(t)$$

then

$$\lim_{t\to+\infty} |x_i(t) - u_i(t)| = 0 \qquad (i = 1, 2, \ldots, n).$$

This completes our proof.

NOTE 2. It follows from the proof of Theorem 3 that the condition

$$a_{jj}(t) > \sum_{i=1, i\ne j}^{n} a_{ij}(t), \qquad (j = 1, 2, \ldots, n)$$

is actually a sufficient one for the global attractivity of any positive periodic solution of (1). So, the local asymptotic stability of positive periodic solution in Theorem 4 of [3] is actually the global one.

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