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GLOBAL ASYMPTOTIC BEHAVIOR IN SOME COOPERATIVE SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to the study of global asymptotic behavior in some cooperative systems of functional differential equations. We first prove a general global result for both discrete-time and continuous dynamical systems on the subset of a strongly ordered Banach space. Then we discuss cooperative systems of functional differential equations and further obtain a threshold theorem on their global asymptotic behavior under the sublinearity assumption. Finally, we give some application examples in epidemic and population dynamics. Some related earlier results are generalized and improved.

1. Introduction. Recently there have been extensive investigations and developments in both continuous and discrete-time monotone dynamical systems (see [**3, 4, 8–11, 14–21**] and references therein). In [**17**], Smith developed sufficient conditions for systems of functional differential equations (FDEs) to generate monotone semiflows (in this case, the system is called the cooperative one) and then obtained some important results on the qualitative behavior of cooperative and irreducible systems of FDEs by applying monotone dynamical system theory. In particular, Smith made an interesting observation (see [**17**, Corollary 3.2]) that the stability type of a steady state for a cooperative system of FDEs is the same as that for its associated cooperative systems of ordinary differential equations (ODEs). Motivated by the consideration of some epidemic and population dynamical models, we are mainly concerned with the global asymptotic behavior of the modeled systems. The convergence and stability in the discrete strongly monotone and sublinear, i.e., subhomogeneous, dynamical systems were studied in [**8, 11, 19**]. In [**11**], the monotone and strongly sublinear,

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i.e., strongly subhomogeneous, dynamical systems were also discussed. In [**21**], one of the authors studied the global attractivity and stability for some discrete strongly monotone dynamical systems, and in particular obtained a threshold result on the global asymptotic stability under the strict sublinearity assumption and gave some applications to time-periodic parabolic equations and cooperative systems of ODEs. The aim of this paper is to develop a different machinery to apply the above mentioned ideas to the cooperative systems of FDEs. Since the semiflow generated by our cooperative systems of FDEs is only monotone and sublinear and the semiflow generated by its corresponding systems of ODE without delays is strictly sublinear, we are unable to use the known abstract results in [**11**] and [**19**]. Furthermore, our abstract results on monotone maps and semiflows may also find their applications to other cooperative systems of differential equations such as FDEs with infinite delay and reaction-diffusion equations with or without delays.

The organization of this paper is as follows. In Section 2 we discuss the abstract discrete-time monotone dynamical systems $\{S^n\}_{n=0}^{\infty}$ and continuous ones $\{T(t)\}_{t\geq 0}$ on the subset V of a strongly ordered Banach space (E, P), and obtain two results on the global asymptotics and existence of positive steady states (Theorems 2.1 and 2.2). Theorem 2.1 with Remark 2.1 generalizes one of Smith's results (see [**16**, Theorem 2.1]) in the sense that we only assume the existence of the Fréchet derivative of S at its fixed point $u = a$ without any other smoothness assumption on S (see Remark 2.4). For the completeness and latter application, we also state three results (Propositions 2.1 and 2.2 and Lemma 2.1), which come from [**21**].

In Section 3 we apply Theorem 2.2 in Section 2 to cooperative systems of FDEs and obtain two global results (Theorems 3.1 and 3.2). Theorem 3.1 has an interesting biological interpretation. That is, for a multi-species cooperative system of FDEs, the system is uniformly persistent if there exists a bounded positive solution (Remark 3.2). Theorem 3.2, which is the main result of this paper, shows that the global asymptotic behavior of cooperative system of FDEs is completely determined by the local stability of zero solution of the associated cooperative system of ODEs. Note that for a cooperative system of ODEs, there are simple tests for the stability of its steady state (see, e.g., [**17**, Corollary] and [**18**, Theorem 2.7]). Clearly there are simple versions of Theorems 3.1 and 3.2 for a cooperative system of ODEs (Corollaries 3.1 and 3.2).

In Section 4, as an illustration, we give some application examples of a threshold theorem in Section 3. The epidemic models with positive feedback [**2**, Section 4.4], the two-species competition chemostat models with delay [**5, 12**], a population model with dispersal and stage structure [**20**] and single-species discrete diffusion systems [**13**] are reconsidered and the related results are generalized and improved further (Propositions 4.1–4.3 and Remarks 4.1–4.3).

2. Global asymptotics in monotone dynamical systems. Let (E, P) be an ordered Banach space with positive cone P. For $x, y \in E$, we write

$$x \geq y \text{ if } x - y \in P$$
$$x > y \text{ if } x - y \in P\backslash\{0\}.$$

If P has nonempty interior $\operatorname{int}(P)$, we also write

$$x \gg y \quad \text{if } x - y \in \operatorname{int}(P).$$

Let V be a subset of E. A continuous mapping $S : V \to E$ is said to be monotone (nondecreasing) if $S(x) \geq S(y)$ for any $x, y \in V$ with $x \geq y$, and strongly monotone if $S(x) \gg S(y)$ for any $x, y \in V$ with $x > y$.

Suppose that X is a complete metric space and $S : X \to X$ is a continuous mapping; then $\{S^n\}_{n=0}^{\infty}$ is called a discrete dynamical system on X. For any $x \in X$, the positive orbit $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \{S^n; n = 0, 1, 2, \dots, \infty\}$.

We first prove the following result.

Theorem 2.1. *Let P be a normal cone with nonempty interior. Assume that*

(1) $S : V = a + P \to V$ is a continuous and monotone mapping and any bounded positive orbit in V is precompact, i.e., for any $u \in V$ for which $\gamma^+(u)$ is bounded, $\overline{\gamma^+(u)}$ is compact in E;

(2) $S(a) = a$, $DS(a)$ is compact and strongly positive, and $r(DS(a)) > 1$, where $DS(a)$ is the Fréchet derivative of S at $u = a$ and $r(DS(a))$ is the spectral radius of linear operator $DS(a) : E \to E$. Then either

(a) *for any* $u > a$, $\lim_{n\to\infty} \|S^n(u)\| = +\infty$, *or alternatively,*

(b) *there exists* $u^* = S(u^*) \gg a$ *such that for any* $a < u \le u^*$, $\lim_{n\to\infty} S^n(u) = u^*$.

Proof. By the Krein-Rutman theorem, see, e.g., [**1**, Chapter I.3] or [**8**, Chapter I.7], $r = r(DS(a))$ is the principal eigenvalue of $DS(a)$. Let $e \gg 0$ be the principal eigenfunction of $DS(a)$ with $\|e\|_E = 1$, i.e., $DS(a)e = re$. For $\varepsilon > 0$,

$$\begin{aligned} S(a+\varepsilon e) &= S(a) + DS(a)(\varepsilon e) + o(\varepsilon) \\ &= a + \varepsilon[re + o(\varepsilon)/\varepsilon]. \end{aligned}$$

Since $r > 1$ and $(r-1)e \in \operatorname{int}(P)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $(r-1)e + o(\varepsilon)/\varepsilon \in \operatorname{int}(P)$, and hence

$$S(a+\varepsilon e) - (a+\varepsilon e) = \varepsilon[(r-1)e + o(\varepsilon)/\varepsilon] \gg 0$$

i.e., for any $\varepsilon \in (0, \varepsilon_0]$, $S(a+\varepsilon e) \gg a + \varepsilon e$. We further have the following claims.

Claim 1. *For any* $u > a$, $S(u) \gg a$.

Indeed, for any given $u > a$, let $u = a + v$, then $v > 0$. For $t > 0$, we have

$$\begin{aligned} S(a+tv) &= S(a) + DS(a)(tv) + o(t) \\ &= a + t(DS(a)v + o(t)/t). \end{aligned}$$

Since $v > 0$ and $DS(a)$ is strongly positive, $DS(a)v \in \operatorname{int}(P)$ and hence there exists $t_0 \in (0,1]$ such that for any $t \in (0, t_0]$, $DS(a)v + o(t)/t \in \operatorname{int}(P)$. Then for any $t \in (0, t_0]$, $S(a+tv) \gg a$. Therefore, by the monotonicity of S, for any $t \in (0, t_0]$, $S(u) = S(a+v) \ge S(a+tv) \gg a$.

Claim 2. *For any* $u > a$ *with* $S(u) = u$, $u \gg a + \varepsilon_0 e$.

In fact, let $\varepsilon_1 = \sup\{\varepsilon \ge 0, u \ge a + \varepsilon e\}$; by Claim 1, $u \gg a$ and hence $\varepsilon_1 > 0$. Assume that, by contradiction, $\varepsilon_1 \le \varepsilon_0$. Since $u \ge a + \varepsilon_1 e$, $u = S(u) \ge S(a + \varepsilon_1 e) \gg a + \varepsilon_1 e$. It follows that there exists $\varepsilon_2 > \varepsilon_1$

such that $u \gg a+\varepsilon_2 e$, which contradicts the definition of ε_1. Therefore $\varepsilon_1 > \varepsilon_0$ and hence $u \geq a+\varepsilon_1 e \gg a+\varepsilon_0 e$.

As shown above, for any $\varepsilon \in (0, \varepsilon_0]$, $S(a+\varepsilon e) \gg a+\varepsilon e$, and then by the monotonicity of S,

$$a+\varepsilon e \ll S(a+\varepsilon e) \leq S^2(a+\varepsilon e) \leq \cdots \\ \leq S^n(a+\varepsilon e) \leq S^{n+1}(a+\varepsilon e) \leq \cdots .$$

Since, by the normality of P, we may assume $\|\cdot\|_E$ is nondecreasing, $\|S^n(a+\varepsilon e)\| \leq \|S^{n+1}(a+\varepsilon e)\|$, $n = 1, 2, \ldots, \infty$. We distinguish between two cases:

(a) for any $\varepsilon \in (0, \varepsilon_0]$, $\{S^n(a+\varepsilon e)\}_{n=1}^{\infty}$ is unbounded. Then $\lim_{n\to\infty} \|S^n(a+\varepsilon e)\| = +\infty$. For any $u > a$, by Claim 1, $S(u) \gg a$. Then there exists $\varepsilon \in (0, \varepsilon_0]$ such that $S(u) \geq a+\varepsilon e$ and hence for all $n = 1, 2, \ldots, \infty$, $S^{n+1}(u) \geq S^n(a+\varepsilon e)$ and $\|S^{n+1}(u)\| \geq \|S^n(a+\varepsilon e)\|$. Therefore, $\lim_{n\to\infty} \|S^n(u)\| = +\infty$.

(b) There exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that $\{S^n(a+\varepsilon_1 e)\}_{n=1}^{\infty}$ is bounded. Then there exists a sufficiently large $\varepsilon^* > 0$ such that for any $n = 1, 2, \ldots, \infty$, $S^n(a+\varepsilon_1 e) \in [a, a+\varepsilon^* e]$. Therefore by the monotonicity of S, for any $\varepsilon \in (0, \varepsilon_1]$ and all $n = 1, 2, \ldots, \infty$, $a+\varepsilon e \ll S(a+\varepsilon e) \leq S^n(a+\varepsilon e) \leq S^n(a+\varepsilon_1 e) \leq a+\varepsilon^* e$ and hence $\|S^n(a+\varepsilon e)\| \leq \|a+\varepsilon^* e\|$. Therefore by the precompactness of $\gamma^+(a+\varepsilon e)$ and monotonicity of $\{S^n(a+\varepsilon e)\}_{n=1}^{\infty}$, for any $\varepsilon \in (0, \varepsilon_1]$,

$$\lim_{n\to\infty} S^n(a+\varepsilon e) = u(\varepsilon), \qquad S(u(\varepsilon)) = u(\varepsilon) \gg a,$$

and clearly, $u(\varepsilon) \leq u(\varepsilon_1)$. For any $\varepsilon \in (0, \varepsilon_1]$, by Claim 2, $u(\varepsilon) \gg a+\varepsilon_0 e \geq a+\varepsilon_1 e$, and hence $u(\varepsilon) \geq S^n(a+\varepsilon_1 e)$, $n = 1, 2, \ldots, \infty$. Therefore $u(\varepsilon_1) = \lim_{n\to\infty} S^n(a+\varepsilon_1 e) \leq u(\varepsilon)$. Then for any $\varepsilon \in (0, \varepsilon_1]$, $u(\varepsilon) = u(\varepsilon_1)$. Let $u^* = u(\varepsilon_1)$; then $u^* \gg a$ and $\lim_{n\to\infty} S^n(a+\varepsilon e) = u^*$. For any $a < u \leq u^*$, by Claim 1, $a \ll S(u) \leq u^*$ and hence there exists $\varepsilon \in (0, \varepsilon_1]$ such that $a+\varepsilon e \leq S(u) \leq u^*$ and $S^n(a+\varepsilon e) \leq S^{n+1}(u) \leq u^*$, $n = 1, 2, \ldots, \infty$. Then, by the normality of P, $\lim_{n\to\infty} S^n(u) = u^*$.

This completes the proof. □

Recall that $S : X \to X$ is asymptotically smooth [**7**, Section 2.2] if for any nonempty closed and bounded set $B \subset X$ for which $SB \subset B$, there exists a compact set $J \subset B$ such that J attracts B, i.e., for any $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon, J, B)$ such that $S^n B$ belongs to the ε-neighborhood of J for $n \geq n_0$. For examples of interesting asymptotic smooth maps, we refer to [**7**, Section 2.3]. It is known that for an asymptotic smooth map, any bounded positive orbit $\gamma^+(x)$ is precompact (see [**7**, Corollary 2.2.4]). Moreover, we have the following remark.

Remark 2.1. If, in addition, $S : V \to V$ is asymptotically smooth, then in the alternative (b) of Theorem 2.1, there exists a monotone entire orbit $\{u_n\}_{n\in Z}$ connecting a and u^*, i.e., $u_{n+1} = S(u_n)$, $u_{n+1} \geq u_n$, $n \in Z, \lim_{n\to-\infty} u_n = a$ and $\lim_{n\to\infty} u_n = u^*$. Indeed, we have shown that there exists a strict subequilibrium $a + \varepsilon e$, $\varepsilon \in (0, \varepsilon_0]$, as close to a as we wish. By the asymptotic smoothness of S, it easily follows that for any $v_k \in B = [a, u^*]$, $k = 1, 2, \ldots, \infty$, and $n_k \to \infty$, $k \to +\infty$, $\{S^{n_k}(v_k)\}_{k=1}^{\infty}$ is precompact. Therefore a careful diagonalization argument given in Dancer-Hess connecting orbit theorem (see [**3**, Proposition 1] or [**8**, Proposition 2.1]) proves the existence of the monotone entire orbit connecting a and u^*.

Remark 2.2. In case $V = a - P$, it is easily seen that there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $S(a - \varepsilon e) \ll a - \varepsilon e$ and hence $a - \varepsilon e$ is a strict superequilibrium. Then the analogous conclusion holds.

Remark 2.3. Let $b > a$ and consider $S : V = [a, b]_E \to V$ (or $b < a$ and $V = [b, a]_E$). Then only the alternative (b) in Theorem 2.1 holds. For more details and a different approach, we refer to [**21**].

Remark 2.4. Theorem 2.1 with Remark 2.1 is very similar to a result due to H.L. Smith [**16**, Theorem 2.1]. Notice that here we only assume the existence of the Fréchet derivative of S at $u = a$ without any other smoothness assumption on S. Moreover, an advantage of Theorem 2.1 (with Remark 2.1), as noticed originally by H.L. Smith, see, e.g., [**16–18**], is that we need only suppose the existence of a single unstable steady state.

A continuous-time semidynamical system (or semiflow) $T(t) : U \subset X \to X$, $t \geq 0$, is called monotone if for any $t \geq 0$ and $x, y \in U$ with $x \geq y$, $T(t)x \geq T(t)y$. As an application of Theorem 2.1, we have the following result.

Theorem 2.2. *Let P be a normal cone with int $(P) \neq \varnothing$. Assume that*

(1) *$T(t) : V = a + P \to V$, $t \geq 0$, is a monotone C^0-semiflow and $T(t)a = a$ for all $t \geq 0$;*

(2) *there exists $t_0 > 0$ such that for $S = T(t_0) : V \to V$, every bounded positive orbit in V is precompact, $DS(a)$ is compact and strongly positive and $r = r(DS(a)) > 1$. Then either*

(a) *for any $u > a$, $\limsup_{t\to\infty} \|T(t)u\| = +\infty$. If, in addition, for any $u \gg a$ and $t \geq 0$, $T(t)u \gg a$, then $\lim_{t\to\infty} \|T(t)u\| = +\infty$, or alternatively,*

(b) *there exists $u^* \gg a$ with $T(t)u^* = u^*$ for all $t \geq 0$ such that for $a < u \leq u^*$, $\lim_{t\to\infty} T(t)u = u^*$.*

Proof. Since $T(t)$, $t \geq 0$, is a semiflow, for any $u \in V$, $S^n(u) = T(nt_0)u$. By applying Theorem 2.1 to $S = T(t_0) : V \to V$, we have precisely two alternatives:

(a) for any $u > a$, $\lim_{n\to\infty} \|S^n(u)\| = \lim_{n\to\infty} \|T(nt_0)u\| = +\infty$. Clearly, $\limsup_{t\to\infty} \|T(t)u\| = +\infty$. By Claim 2 in the proof of Theorem 2.1, for any $u > a$, $T(t_0)u \gg a$ and hence by our additional assumption for any $t \geq 0$, $T(t + t_0)u = T(t)(T(t_0)u) \gg a$, i.e., for any $t \geq t_0$, $T(t)u \gg a$. Then by the continuity of $T(t)u$ with respect to t on the compact set $[t_0, 2t_0]$, for a given $e \in \text{int}\,(P)$, there exists $\varepsilon > 0$ such that for any $t \in [t_0, 2t_0]$, $T(t)u \gg a + \varepsilon e$. For any $t \geq t_0$, let $t = nt_0 + t'$, where $n = [(t - t_0)/t_0]$ is the greatest integer less than or equal to $(t - t_0)/t_0$ and $t' \in [t_0, 2t_0)$. Then $T(t')u \gg a + \varepsilon e$ and $T(t)u = T(nt_0)(T(t')u) \geq T(nt_0)(a + \varepsilon e)$. By the normality of the cone P, we may assume $\|\cdot\|$ is nondecreasing. Then $\|T(t)u\| \geq \|T(nt_0)(a + \varepsilon e)\| = \|S^n(a + \varepsilon e)\|$ and hence $\lim_{t\to\infty} \|T(t)u\| = +\infty$.

(b) there exists $u^* = T(t_0)u^* \gg a$ such that for any $a < u \leq u^*$, $\lim_{n\to\infty} T(nt_0)u = u^*$. Clearly, $T(t)u^*$ is t_0-periodic with respect to t and $\gamma^+(u^*) = \{T(t)u^*; t \geq 0\}$ is compact. For any $a < u \leq u^*$, by the continuity of $T(t)u$ with respect to $u \in V$ uniformly for t on the compact set $[0, t_0]$, it easily follows that $\lim_{t\to\infty}(T(t)u - T(t)u^*) = 0$. Therefore $\gamma^+(u) = \{T(t)u; t \geq 0\}$ is precompact. For any given $\varepsilon \in (0,1)$, let $u_\varepsilon = a + \varepsilon(u^* - a)$; then $a \ll u_\varepsilon \ll u^*$ and $\gamma^+(u_\varepsilon)$ is precompact. Since $\lim_{n\to\infty} T(nt_0)u_\varepsilon = u^*$, there exists $N > 0$ such that $T(nt_0)u_\varepsilon \gg u_\varepsilon$ for all $n \geq N$. By the Hirsch convergence criterion for monotone flows (see [**9**, Theorem 2.3] or [**10**, Theorem 6.4]), $T(t)u_\varepsilon$ converges to an equilibrium as $t \to \infty$, which implies that u^* is an equilibrium, i.e., $T(t)u^* = u^*$ for all $t \geq 0$. Therefore, for any $a < u \leq u^*$,

$$0 = \lim_{t\to\infty}(T(t)u - T(t)u^*) = \lim_{t\to\infty}(T(t)u - u^*),$$

that is, $\lim_{t\to\infty} T(t)u = u^*$.

This completes the proof. □

Let either $V = [a, b]_E$ with $b > a$ or $V = a + P$ and let $S : V \to V$ be a continuous and monotone mapping with $S(a) = a$. As claimed in the proof of Theorem 2.1, for any $u \in V$ with $u > a$, $S(u) \gg a$. By a careful examination of the proof of [**21**, Theorem 2.2], we then have the following result.

Proposition 2.1. *Let either $V = [0, b]_E$ with $b > 0$ or $V = P$, and let $S : V \to V$ be a continuous and monotone mapping with the property that any bounded positive orbit in V is precompact. Assume that*

(1) *$S(0) = 0$, $DS(0)$ is compact and strongly positive, and $r(DS(0)) \leq 1$;*

(2) *$S(u) < DS(0)u$ for any $u \in V$ and $u \gg 0$.*

Then $u = 0$ is globally attractive with respect to V.

Let either $V = [0, b]_E$ with $b > 0$ or $V = P$. According to [**21**], a mapping $S : V \to E$ is called strictly sublinear on V if $S(\alpha u) > \alpha S(u)$ for all $\alpha \in (0,1)$ and $u \in V$ with $u \gg 0$. From the proof of [**21**, Lemma 1], we have the following result.

Lemma 2.1. *Let either $V = [0, b]_E$ with $b > 0$ or $V = P$. Assume* *that $S : V \to V$ is continuous and strictly sublinear on V. If $S(0) = 0$ and $DS(0)$ exists, then $S(u) < DS(0)u$ for all $u \in V$ with $u \gg 0$.*

Recall that $S : V \subset E \to E$ is called a strongly monotone mapping if $S(u) \gg S(v)$ for all $u, v \in V$ with $u > v$. For a strongly monotone mapping with strict sublinearity, we have the following result on the uniqueness of positive fixed points, which also comes from [**21**, Lemma 1].

Proposition 2.2. *Let either $V = [0, b]_E$ with $b > 0$ or $V = P$. Assume that $S : V \to V$ is continuous, strongly monotone and strictly sublinear on V. Then S admits at most one positive fixed point in V.*

As remarked in [**21**, Remark 2.2], both Smith's concavity assumption in [**15**], i.e., $DS(v) - DS(u) > 0$ if $u \gg v \gg 0$, and Krasnoselskii's strong concavity, i.e., for every $u \gg 0$ and $\alpha \in (0,1)$, there exists $\eta = \eta(u, \alpha) > 0$ such that $S(\alpha u) \geq (1 + \eta)\alpha S(u)$, imply our strict sublinearity.

3. Applications to cooperative systems of FDEs. According to [**17**], let $r = (r_1, \dots, r_n) \in R_n^+ = \{(x_1, \dots, x_n); x_i \geq 0, i = 1, 2, \dots, n\}$, $|r| = \max_{1 \leq i \leq n} r_i$, and define $C_r = \prod_{i=1}^n C([-r_i, 0], R)$. For $\phi = (\phi_1, \dots, \phi_n) \in C_r$, define $\|\phi\| = \sum_{i=1}^n \|\phi_i\|_\infty$, where $\|\phi_i\|_\infty = \max_{\theta \in [-r_i, 0]} |\phi_i(\theta)|$. Then C_r is a Banach space. Let $C_r^+ = \{(\phi_1, \phi_2, \dots, \phi_n) \in C_r; \phi_i(\theta) \geq 0, 1 \leq i \leq n, \theta \in [-r_i, 0]\}$, then C_r^+ is a normal cone of C_r with nonempty interior in C_r. Let $\wedge$ denote the inclusion $R^n \to C_r$ by $x \to \hat{x}$, $\hat{x}_i(\theta) \equiv x_i$, $\theta \in [-r_i, 0]$, $i = 1, 2, \dots, n$. Given function $x_i(t) \in C_r$ defined on $[-r_i, \sigma), \sigma > 0$, and $0 \leq t < \sigma$, define $x_t \in C_r$ by $x_t = (x_t^1, \dots, x_t^n)$, where $x_t^i(\theta) = x_i(t + \theta)$, $\theta \in [-r_i, 0]$, $i = 1, 2, \dots, n$.

Consider the functional differential equations

$$dx(t)/dt = f(x_t) \tag{3.1}$$

where $f : U \to R^n$ is a continuously differentiable map on the open subset U of C_r. We write $x(t, \phi)$ for the unique solution of (3.1) satisfying $x_0 = \phi$ and let $[0, \sigma_\phi)$ be its maximal interval of existence.

Assume that U is order convex, that is, if $\phi, \psi \in U$ with $\phi \leq \psi$, then $s\phi + (1-s)\psi \in U$ for $0 \leq s \leq 1$.

Definition. f is called *cooperative* in U if, for any $\psi \in U$, $L = df(\psi)$ satisfies

(K) For all $\phi \in C_r^+$ with $\phi_i(0) = 0$, $1 \leq i \leq n$, $L_i(\phi) \geq 0$.

Let $L : C_r \to R^n$ be a bounded linear map. Then $L = (L_1, \dots, L_n)$ admits the following standard representation (see [**17**])

$$L_i(\phi) = \sum_{j=1}^{n} \int_{-r_j}^{0} \phi_j(\theta)\, d\eta_{ij}(\theta)$$

where $\eta_{ij} : R \to R$ satisfies

(i) $\eta_{ij}(\theta) = \eta_{ij}(0)$ for $\theta \geq 0$, $\eta_{ij}(\theta) = 0$ for $\theta \leq -r_j$;

(ii) η_{ij} is a bounded variation on $[-r_j, 0]$ and η_{ij} is continuous from the left on $(-r_j, 0)$.

Define the stability modulus of L as

$$s(L) = \max\{\operatorname{Re}\lambda; \det \Delta(\lambda) = 0\}$$

where $\Delta(\lambda) = \lambda I - A(\lambda)$ and $(A(\lambda))_{ij} = \int_{-r_j}^{0} e^{\lambda\theta} d\eta_{ij}(\theta)$, $1 \leq i, j \leq n$. We will make the following assumption.

(R) For each j for which $r_j > 0$, there exists an i such that $\eta_{ij}(\theta) > 0$ for all $\theta \in (-r_j, 0)$.

Now we are in a position to prove the following result.

Theorem 3.1. *Let $f : C_r^+ \to R^n$ be a continuously differentiable cooperative map with the property that f maps bounded sets into bounded sets and define $F : R_+^n \to R^n$ by $F(x) = f(\hat{x})$, $x \in R_+^n$. Assume that*

(1) *for any $\phi \in C_r^+$ with $\phi_i(0) = 0$, $f_i(\phi) \geq 0$ and $\sigma_\phi = +\infty$ for any $\phi \in C_r^+$;*

(2) *$f(\hat{0}) = 0$, $DF(0)$ is irreducible and $L = df(\hat{0})$ satisfies* (R);

(3) *$s(DF(0)) = \max\{\operatorname{Re}\lambda; \det(\lambda I - DF(0))\} > 0$.*

Then either

(a) *for any* $\phi \in C_r^+ \backslash \{\hat{0}\}$, $\lim_{t\to\infty} \|x_t(\phi)\| = +\infty$, *or alternatively,*

(b) *there exists* $x^* \gg 0$ (*in* R^n) *with* $F(x^*) = 0$ *such that for any* $\hat{0} < \phi \leq \hat{x}^*$, $\lim_{t\to\infty} x(t,\phi) = x^*$. *Moreover, for any* $\phi > \hat{0}$, $\liminf_{t\to\infty} x(t,\phi) \geq x^*$, *where* $\liminf_{t\to\infty} x(t,\phi) = (\liminf_{t\to\infty} x_1(t,\phi), \dots, \liminf_{t\to\infty} x_n(t,\phi))$.

Proof. For any $\phi \in C_r^+$, by assumption (1), $x(t,\phi)$ exists globally on $[0,+\infty)$, and by [**17**, Proposition 1.2], $x(t,\phi) \geq 0$, $t \geq 0$. Define $S(t)\phi = x_t(\phi)$, $\phi \in C_r^+$, $t \geq 0$. Since f is cooperative in C_r^+, by [**17**, Lemma 2.4] and comparison theorem of FDEs, see, e.g., [**17**, Proposition 1.1], $S(t) : C_r^+ \to C_r^+$ is a monotone C^1-semiflow and $S(t)\hat{0} = \hat{0}$ for $t \geq 0$. Moreover, $S(t) : C_r^+ \to C_r^+$ is conditionally completely continuous for $t \geq |r|$, see [**7**, Theorem 4.1.1]. We further claim that for any $\phi, \psi \in C_r^+$ with $\psi \gg \phi$, $S(t)\psi \gg S(t)\phi$, $t \geq 0$. Indeed,

$$x(t,\psi) - x(t,\phi) = \int_0^1 d_\phi x(t, s\psi + (1-s)\phi)(\psi - \phi)\, ds.$$

For each fixed $s \in [0,1]$, let $\beta = \psi - \phi$, $\xi = s\psi + (1-s)\phi$, by [**6**, Theorem 4.1], $d_\phi x(t,\xi)\beta = y(t,\beta)$ satisfies the linear variational equation

$$dy(t)/dt = df(x_t(\xi))y_t, \qquad y_0 = \beta. \tag{3.2}$$

Since $\beta \gg 0$, f is cooperative in C_r^+ and hence $L(t,\cdot) = df(x_t(\xi))$ satisfies (K) for each $t \geq 0$, by [**17**, Lemma 2.1], $y(t,\beta) \gg 0$ for all $t \geq 0$. Then for any $s \in [0,1]$ and $t \geq 0$, $d_\phi x(t, s\psi + (1-s)\phi)(\psi - \phi) \gg 0$ and hence for all $t \geq 0$, $x(t,\psi) - x(t,\phi) \gg 0$. Then $S(t)\psi \gg S(t)\phi$, $t \geq 0$. In particular, for any $\phi \gg \hat{0}$, $t \geq 0$, $S(t)\phi \gg S(t)\hat{0} = \hat{0}$.

Now consider the linear variational equation of (3.1) about the steady state $x = \hat{0}$:

$$dz(t)/dt = df(\hat{0})z_t. \tag{3.3}$$

For any $\phi \in C_r$, let $z(t,\phi)$ be the unique solution of (3.3) satisfying $z_0 = \phi$; then $z(t,\phi)$ exists globally on $[0,+\infty)$. Define $T(t) : C_r \to C_r$, $t \geq 0$, by $T(t)\phi = z_t(\phi)$; then $T(t)$ is a compact linear operator for $t \geq |r|$ and $r(T(t))$, the spectral radius of $T(t)$, satisfies $r(T(t)) =$

e^{ts}, $s = s(df(\hat{0}))$, see [**6**]. Since $df(\hat{0})$ satisfies (K) and (R), and $DF(0) = df(\hat{0})(\hat{e}_1, \dots, \hat{e}_n)$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis in R^n, by [**17**, Proposition 2.3], $T(t)(C_r^+ \backslash \{\hat{0}\}) \subset \text{int}\ (C_r^+)$ for $t \geq (n+1)|r|$, i.e., $T(t)$ is strongly positive for all $t \geq (n+1)|r|$. Again by [**6**, Theorem 4.1], for any $\psi \in C_r$, $d_\phi x(t, \hat{0})\psi = z(t, \psi)$, $t \geq 0$. It easily follows that for any $t \geq |r|$ and any $\psi \in C_r$, $Dx_t(\hat{0})\psi = z_t(\psi) = T(t)\psi$, i.e., $Dx_t(\hat{0}) = T(t)$ for all $t \geq |r|$. Then for any given $t_0 \geq (n+1)|r|$, $S = S(t_0) = x_{t_0} : C_r^+ \to C_r^+$ is conditionally completely continuous and hence asymptotically smooth (see [**7**, Lemma 2.3.1]), $DS(\hat{0}) = Dx_{t_0}(\hat{0}) = T(t_0)$ is compact and strongly positive, and since $s(dF(0)) \cdot s(df(\hat{0})) > 0$ by [**17**, Corollary 3.2], $r(DS(\hat{0})) = e^{t_0 \cdot s(df(\hat{0}))} > 1$.

By Theorem 2.2 in Section 2, it follows that either

(a) for any $\phi > 0$, $\lim_{t\to\infty} \|S(t)\phi\| = \lim_{t\to\infty} \|x_t(\phi)\| = +\infty$, or alternatively,

(b) there exists $x^* \gg 0$ with $f(\hat{x}^*) = 0$ such that for any $\hat{0} < \phi \leq \hat{x}^*$, $\lim_{t\to\infty} x_t(\phi) = \hat{x}^*$.

Clearly, in case (b), $\lim_{t\to\infty} x(t,\phi) = x^*$. For any $\phi > \hat{0}$, since $DS(t_0)(\hat{0})$ is strongly positive, $S(t_0)\phi \gg 0$ (see, e.g., Claim 1 in the proof of Theorem 2.1), there exists $0 < \varepsilon_0 < 1$ such that $\varepsilon_0 \hat{x}^* \leq S(t_0)\phi = x_{t_0}(\phi)$ and hence

$$\begin{aligned}\liminf_{t\to\infty} x(t + t_0, \phi) &= \liminf_{t\to\infty} x(t, x_{t_0}(\phi)) \\ &\geq \lim_{t\to\infty} x(t, \varepsilon_0 \hat{x}^*) = x^*.\end{aligned}$$

That is, $\liminf_{t\to\infty} x(t,\phi) \geq x^*$.

This completes the proof. □

Remark 3.1. A similar conclusion was obtained in [**17**, Theorem 3.3] under the stronger assumption that f is cooperative and irreducible in C_r^+ and that df is Lipschitz continuous in a neighborhood of $\hat{0}$.

Remark 3.2. Theorem 3.1 has an interesting biological interpretation if (3.1) describes a multi-species cooperative system. That is, the system is uniformly persistent if there exists a bounded positive solution.

Now we are ready to prove the main result of this paper, which provides a threshold criterion on the global asymptotics of system (3.1) under some appropriate assumptions.

Theorem 3.2. *Let $f : C_r^+ \to R^n$ be a continuously differentiable cooperative map, and let $F : R_+^n \to R^n$ be defined by $F(x) = f(\hat{x})$, $x \in R_+^n$. Assume that*

(1) *for any $\phi \in C_r^+$ with $\phi_i(0) = 0$, $f_i(\phi) \geq 0$ and f maps bounded sets into bounded sets;*

(2) *$f : C_r^+ \to R^n$ is sublinear, i.e., for any $\alpha \in (0,1)$, $\phi \in C_r^+$, $f(\alpha\phi) \geq \alpha f(\phi)$, and $F : R_+^n \to R$ is strictly sublinear, i.e., for any $\alpha \in (0,1)$, $x \in R_+^n$ with $x \gg 0$, $F(\alpha x) > \alpha F(x)$;*

(3) *$f(\hat{0}) = 0$, $df(\hat{0})$ satisfies* (R), *and for any $x \in R_+^n$, $DF(x)$ is irreducible.*

(a) *If $s(DF(0)) \leq 0$, then $\hat{0}$ is globally asymptotically stable for* (3.1) *with respect to C_r^+;*

(b) *If $s(DF(0)) > 0$, then either*

(i) *for any $\phi \in C_r^+ \backslash \{\hat{0}\}$, $\lim_{t\to\infty} \|x_t(\phi)\| = +\infty$, or alternatively,*

(ii) (3.1) *admits a unique positive steady state $\hat{x}^*$ with $x^* \gg 0$ and $\hat{x}^*$ is globally asymptotically stable with respect to $C_r^+ \backslash \{\hat{0}\}$.*

Proof. We first prove that, for any $\phi \in C_r^+$, $\sigma_\phi = [0, +\infty)$, i.e., $x(t,\phi)$ exists globally on $[0,\infty)$. Since $f(\hat{0}) = 0$, the sublinearity of f implies that for any $\phi \geq 0$, $f(\phi) = \lim_{\alpha\to 0^+} \alpha f(\phi)/\alpha \leq \lim_{\alpha\to 0^+}(f(\alpha\phi) - f(\hat{0}))/\alpha = Df(\hat{0})\phi$. By assumption (1), for any $\phi \in C_r^+$, $x(t,\phi) \geq 0$, $t \in [0, \sigma_\phi)$. Then

$$dx(t,\phi)/dt = f(x_t(\phi)) \leq Df(\hat{0})x_t(\phi).$$

Let $y(t,\phi)$ be the unique solution of linear FDE

$$\begin{cases} dy(t)/dt = Df(\hat{0})y_t \\ y_0 = \phi; \end{cases}$$

then $y(t,\phi)$ exists globally on $[0, +\infty)$. By comparison theorem of FDEs, see, e.g., [**17**, Proposition 1.1], $x(t,\phi) \leq y(t,\phi), t \in [0, \sigma_\phi)$, which implies $\sigma_\phi = +\infty$.

We further claim that when $s(DF(0)) \le 0$, (3.1) admits no steady state in $C_r^+\backslash\{\hat{0}\}$ and that when $s(DF(0)) > 0$, (3.1) admits at most one positive steady state in C_r^+. Indeed, it suffices to prove the corresponding claim for the associated nonlinear ODE

$$dx(t)/dt = F(x(t)). \tag{3.4}$$

From assumptions (1), (2) and (3), it easily follows that (3.4) generates a strongly monotone semiflow $\varphi(t) : R_+^n \to R_+^n$ by $\varphi(t)x = \varphi(t,x)$, $t \ge 0$, where $\varphi(t,x)$ is the unique solution of (3.4) with $\varphi(0,x) = x$, see, e.g., [**10**] or [**18**]. For any given $\omega > 0$, we view (3.4) as an ω-periodic system of ODE. As shown in [**21**], the strict sublinearity of $F(x)$ on R_+^n implies the strict sublinearity of ω-time map $S = \varphi(\omega) : R_+^n \to R_+^n$, which is clearly compact and strongly monotone. Notice that $S(0) = 0$ and $r(DS(0)) = e^{\omega s}$, $s = s(DF(0))$. Now Lemma 2.1 and Propositions 2.1 and 2.2 imply that when $s(DF(0)) \le 0$, S admits no fixed point in $R_+^n\backslash\{0\}$, and that when $s(DF(0)) > 0$, S admits at most one fixed point in $R_+^n\backslash\{0\}$. Clearly, every steady state of (3.4) in R_+^n is a fixed point of ω-time map S. This then proves our claim on the steady state of (3.4) and hence of (3.1).

Define $\Phi(t)\phi = x_t(\phi)$, $\phi \in C_r^+$, $t \ge 0$. Then $\Phi(t) : C_r^+ \to C_r^+$ is a monotone semiflow, see [**17**], and $\Phi(t)\hat{0} = \hat{0}$ for $t \ge 0$. Moreover, for any $t \ge 0$, $\Phi(t)$ is sublinear on C_r^+, i.e., for any $\alpha \in (0,1)$ and any $\phi \in C_r^+$, $\Phi(t)(\alpha\phi) \ge \alpha\Phi(t)(\phi)$, and hence, since $\Phi(t)(\hat{0}) = \hat{0}$, $\Phi(t)\phi \le D\Phi(t)(\hat{0})\phi$. Indeed, for any $\phi \in C_r^+$ and $\alpha \in (0,1)$, let $w(t) = x(t,\alpha\phi) - \alpha x(t,\phi)$, then $w(t)$ satisfies

$$dw(t)/dt = L(t, w_t) + h(t)$$

where

$$L(t,\phi) = \int_0^1 df(sx_t(\alpha\phi) + (1-s)\alpha x_t(\phi))\phi \, ds$$

and

$$h(t) = f(\alpha x_t(\phi)) - \alpha f(x_t(\phi)).$$

Since $w_0 = x_0(\alpha\phi) - \alpha x_0(\phi) = \hat{0} \in C_r^+$, by variation of constants formula [**6**, Theorem 2.1],

$$w(t) = \int_0^t U(t,s)h(s)\, ds, \quad t \ge 0,$$

where $U(t,s)$ is the fundamental matrix defined by linear FDE

$$dw(t)/dt = L(t, w_t).$$

Since f is cooperative, $L(t,\phi)$ satisfies that for any $\phi \in C_r^+$ with $\phi_i(0) = 0$, $L_i(t,\phi) \geq 0$ for $t \geq 0$. Now it easily follows that $U(t,s) \geq 0$ for $t \geq s$. Therefore, for any $t \geq 0$, $w(t) \geq 0$, and hence, since $w_0 = \hat{0}$, $w_t \geq \hat{0}$, i.e., $\Phi(t)(\alpha\phi) = x_t(\alpha\phi) \geq \alpha x_t(\phi) = \alpha\Phi(t)(\phi)$. By Remark 1 in [**17**, Section 3], there exists $u \gg 0$ in R^n such that $y(t) = ue^{s(df(\hat{0}))t}$ is a solution of the variational equation

$$\frac{dy}{dt} = df(\hat{0})y_t.$$

Let $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)$ be defined by

$$\bar{u}_i(\theta) = u_i e^{s(df(\hat{0}))\theta}, \quad -r_i \leq \theta \leq 0, \quad 1 \leq i \leq n;$$

then $\bar{u} \gg 0$ in C_r and $y(t) = y(t,\bar{u})$. Let $T(t)u = y_t(u)$, $t \geq 0$, $u \in C_r^+$; then, as shown before, for any $t \geq |r|$, $D\Phi(t)(\hat{0}) = T(t)$.

In case $s(DF(0)) \leq 0$, since, by [**17**, Corollary 3.2], $s(df(\hat{0})) \leq 0$, for any $t \geq |r|$, $D\Phi(t)(\hat{0})\bar{u} = T(t)\bar{u} = y_t(\bar{u}) \leq \bar{u}$. For any $\beta > 0$, given $t_0 \geq |r|$, $\Phi(t_0)(\beta\bar{u}) \leq D\Phi(t_0)(\hat{0})(\beta\bar{u}) \leq \beta\bar{u}$. By the asymptotic smoothness of $\Phi(t_0)$, $(\Phi(t_0))^n(\beta\bar{u}) = \Phi(nt_0)(\beta\bar{u}) \to u^*$, $n \to \infty$, and $u^* = \Phi(t_0)u^*$. It easily follows that $\Phi(t)u^*$ is t_0-periodic with respect to $t \geq 0$ and $\lim_{t\to\infty} \|\Phi(t)(\beta\bar{u}) - \Phi(t)(u^*)\| = 0$. Then the ω-limit set of $\beta\bar{u}$ for semiflow $\Phi(t)$ is a t_0-periodic orbit $\gamma^+(u^*) = \{\Phi(t)u^*; t \geq 0\}$, i.e., $\omega(\beta\bar{u}) = \gamma^+(u^*)$. Since $t_0 \geq |r|$ is arbitrary, $\omega(\beta\bar{u})$ has any $t_0 \geq |r|$ as a period, which implies that $\omega(\beta\bar{u})$ is a steady state of $\Phi(t)$. By the nonexistence of steady state in $C_r^+\backslash\{\hat{0}\}$, $\omega(\beta\bar{u}) = \hat{0}$, i.e., $\lim_{t\to\infty}\Phi(t)(\beta\bar{u}) = \hat{0}$. Therefore, for any $u \geq \hat{0}$, there exists $\beta > 0$ such that $\hat{0} \leq u \leq \beta\bar{u}$ and hence $\hat{0} \leq \Phi(t)(u) \leq \Phi(t)(\beta\bar{u})$. Then $\lim_{t\to\infty} x_t(u) = \lim_{t\to\infty}\Phi(t)u = \hat{0}$.

In case $s(df(\hat{0})) > 0$, by Theorem 3.1, either

(i) for any $\phi \in C_r^+\backslash\{\hat{0}\}$, $\lim_{t\to\infty}\|x_t(\phi)\| = +\infty$, or alternatively,

(ii) there exists $x^* \gg 0$ in R^n with $F(x^*) = 0$ such that for any $\hat{0} < \phi \leq \hat{x}^*$, $\lim_{t\to\infty} x_t(\phi) = \hat{x}^*$.

In the latter case, for any $\beta > 1$, given $t_0 \geq |r|$, by the sublinearity of $\Phi(t)$, $t \geq 0$,

$$\hat{x}^* \leq \Phi(t_0)(\beta\hat{x}^*) \leq \beta\Phi(t_0)(\hat{x}^*) = \beta\hat{x}^*.$$

By a similar argument to that in the case (a), it follows that $\Phi(t)(\beta\hat{x}^*)$ converges to a steady state u^* and $u^* \geq \hat{x}^* \gg 0$. Then the uniqueness of steady state in $C_r^+\backslash\{\hat{0}\}$ implies that $\lim_{t\to\infty}\Phi(t)(\beta\hat{x}^*) = \hat{x}^*$. Therefore for any $\phi \geq \hat{x}^*$, there exists $\beta > 1$ such that $\hat{x}^* \leq \phi \leq \beta\hat{x}^*$ and hence $x^* \leq \Phi(t)\phi \leq \Phi(t)(\beta\hat{x}^*)$. Then $\lim_{t\to\infty}\Phi(t)\phi = \hat{x}^*$. Since for any $\phi > \hat{0}$, there exist $\hat{0} < \phi_1 \leq \hat{x}^*$ and $\hat{x}^* \leq \phi_2$ such that $\phi_1 \leq \phi \leq \phi_2$ and hence $\Phi(t)(\phi_1) \leq \Phi(t)(\phi) \leq \Phi(t)(\phi_2)$. It follows that $\lim_{t\to\infty}\Phi(t)(\phi) = \hat{x}^*$.

It remains to prove the stability of the steady states of $\hat{0}$ and $\hat{x}^*$ with respect to C_r^+ in case (a) and $C_r^+\backslash\{\hat{0}\}$ in case (ii) respectively. Here we only prove the former and the latter can be proved similarly. For any $\varepsilon > 0$, let $B(\hat{0},\varepsilon) = \{u \in C_r; \|u\| < \varepsilon\}$; there exists $\hat{u} \gg \hat{0}$ such that $[\hat{0},\hat{u}] \subset B(\hat{0},\varepsilon)$. Then for any $t \geq 0$, $\Phi(t)\hat{u} \gg \hat{0}$. Since $\lim_{t\to\infty}\Phi(t)\hat{u} = \hat{0}$, there exists $t_0 > 0$ such that for all $t \geq t_0$, $\Phi(t)\hat{u} \in [\hat{0},\hat{u}]$. Then for any $u \in V_0 = [\hat{0},\Phi(t_0)\hat{u}]] = \{u \in C_r; \hat{0} \leq u \ll \Phi(t_0)\hat{u}\}$, and for any $t \geq 0$,

$$\Phi(t)u \in [\hat{0},\Phi(t+t_0)\hat{u}]] \subset [0,\hat{u}] \subset B(\hat{0},\varepsilon).$$

Since $V = [[-\Phi(t_0)\hat{u},\Phi(t_0)\hat{u}]] = \{u \in C_r; -\Phi(t_0)\hat{u} \ll u \ll \Phi(t_0)\hat{u}\}$ is open in C_r and $\hat{0} \in V$, there exists $\delta > 0$ such that $B(\hat{0},\delta) \subset V$. Then for any $u \in B(\hat{0},\delta) \cap C_r^+ \subset V_0$, $\Phi(t)u \in B(\hat{0},\varepsilon)$, $t \geq 0$. Therefore $u = \hat{0}$ is stable with respect to C_r^+.

This completes the proof. □

For autonomous systems of ODEs,

$$dx(t)/dt = f(x), \quad x \in R^n \tag{3.5}$$

let $E = R^n$ and $P = R_+^n$; then Theorems 3.1 and 3.2 imply the following results respectively.

Corollary 3.1. *Let $f : R_+^n \to R^n$ be a continuously differentiable map. Assume that*

(1) *f is cooperative on R_+^n, i.e., for any $x \in R_+^n$, $\partial f_i/\partial x_j \geq 0$, $i,j = 1,2,\ldots,n$, and $i \neq j$, and $Df(0) = (\partial f_i(0)/\partial x_j)_{1\leq i,j\leq n}$ is irreducible*;

(2) *$f(0) = 0$, $f_i(x) \geq 0$ for all $x \in R_+^n$ with $x_i = 0$, $i = 1,2,\ldots,n$, and for any $x \in R_+^n$, the unique solution $\varphi(t,x)$ of (3.5) with $\varphi(0,x) = x$ exists globally on $[0,+\infty)$*;

(3) $s(Df(0)) = \max\{\operatorname{Re}\lambda; \det(\lambda - Df(0))\} > 0$.

Then either

(a) *for any* $x \in R^n_+ \backslash \{0\}$, $\lim_{t\to\infty} |\varphi(t,x)| = +\infty$, *or alternatively,*

(b) *there exists* $x^* \gg 0$ *with* $f(x^*) = 0$ *such that for any* $0 < x \leq x^*$, $\lim_{t\to\infty} \varphi(t,x) = x^*$. *Moreover, for any* $x > 0$, $\liminf_{t\to\infty} \varphi(t,x) \geq x^*$.

Corollary 3.2. *Let* $f : R^n_+ \to R^n$ *be a continuously differentiable map. Assume that*

(1) f *is cooperative on* R^n_+ *and* $Df(x) = (\partial f_i/\partial x_j)_{1\leq i,j\leq n}$ *is irreducible for every* $x \in R^n_+$;

(2) $f(0) = 0$ *and* $f_i(x) \geq 0$ *for all* $x \in R^n_+$ *with* $x_i = 0$, $i = 1, 2, \ldots, n$;

(3) f *is strictly sublinear on* R^n_+, *i.e., for any* $\alpha \in (0,1)$ *and any* $x \gg 0$, $f(\alpha x) > \alpha f(x)$.

(a) *If* $s(Df(0)) \leq 0$, *then* $x = 0$ *is globally asymptotically stable with respect to* R^n_+;

(b) *If* $s(Df(0)) > 0$, *then either*

(i) *for any* $x \in R^n_+ \backslash \{0\}$, $\lim_{t\to\infty} |\varphi(t,x)| = +\infty$, *or alternatively,*

(ii) (3.5) *admits a unique positive steady state* $x^* \gg 0$ *and* $x = x^*$ *is globally asymptotically stable with respect to* $R^n_+ \backslash \{0\}$.

Remark 3.3. Corollary 3.2 generalizes a similar result of Smith's (see [**15**, Theorem 3.1 and Corollary 3.2]) in the sense that the concavity assumption is weakened into strict sublinearity but the conclusion of the global attractivity of equilibrium is strengthened into its global asymptotic stability.

4. Some examples. In this section we will apply our main results of Section 3 to some epidemic models with a positive feedback, competition models in chemostat with time delay and population models with dispersal and stage structure. Some known results are not only rediscovered but also improved further.

Example 1. Consider the epidemic models with positive feedback (see [**2**, Section 4.4]):

$$\begin{cases} dz_1/dt = -a_{11}z_1 + a_{12}z_2 \\ dz_2/dt = g(z_1) - a_{22}z_2 \end{cases} \tag{4.1}$$

where $a_{11} > 0$, $a_{12} > 0$ and $a_{22} > 0$, and $g : R_+ \to R_+$ is a continuously differentiable function. We have the following result.

Proposition 4.1. *Let $\theta = (a_{12}g'(0))/(a_{11}a_{22})$ and assume that*

(i) $g(0) = 0$ *and* $g'(z) > 0$ *for* $z \geq 0$;

(ii) $g(z)$ *is strictly sublinear on* R_+, *i.e., for any* $z > 0$ *and any* $\alpha \in (0,1)$, $g(\alpha z) > \alpha g(z)$.

(a) *If* $0 < \theta \leq 1$, *then* 0 *is globally asymptotically stable for system* (4.1) *in* R^2_+;

(b) *If* $\theta > 1$ *and for all* $z_1 > 0$, $g(z_1)/z_1 > a_{11}a_{22}/a_{12}$, *then for any* $z \in R^2_+\backslash\{0\}$, $\lim_{t\to\infty}|\varphi(t,x)| = +\infty$, *where* $\varphi(t,z)$ *is the unique solution of* (4.1) *with* $\varphi(0,z) = z$;

(c) *If* $\theta > 1$ *and there exists* $\bar{z}_1 > 0$ *such that* $g(\bar{z}_1)/\bar{z}_1 \leq a_{11}a_{22}/a_{12}$, *then* (4.1) *admits a unique positive equilibrium* $z^* \gg 0$ *and* z^* *is globally asymptotically stable in* $R^2_+\backslash\{0\}$.

Proof. Let

$$f(z) = \begin{pmatrix} -a_{11}z_1 + a_{12}z_2 \\ g(z_1) - a_{22}z_2 \end{pmatrix}.$$

It easily follows that $\theta \leq 1$ implies $s(Df(0)) \leq 0$ and that $\theta > 1$ implies $s(Df(0)) > 0$. Clearly, if $g(z_1)/z_1 > a_{11}a_{22}/a_{12}$ for all $z_1 > 0$, (4.1) admits no positive equilibrium. If $\theta > 1$, i.e., $g'(0) = \lim_{z\to 0^+} g(z)/z > a_{11}a_{22}/a_{12}$, and there exists $\bar{z}_1 > 0$ such that $g(\bar{z}_1)/\bar{z}_1 \leq a_{11}a_{22}/a_{12}$, then there exists $z_1^* > 0$ such that $g(z_1^*)/z_1^* = a_{11}a_{22}/a_{12}$ and hence (4.1) admits a positive equilibrium $z^* \gg 0$. Now Corollary (3.2) completes the proof. $\square$

Remark 4.1. A conclusion similar to that of (a) and (c) in Proposition 4.1 is claimed in [**2**, Theorem 4.1] under the assumption that $g''(z) > 0$ for all $z > 0$ and $\limsup_{t\to+\infty} g(z)/z < a_{11}a_{22}/a_{12}$, which implies clearly the strict sublinearity and the additional assumption in (c). Moreover, Proposition 4.1 also gives a trichotomy on the global asymptotic behavior of system (4.1).

Remark 4.2. By using Corollary 3.2 one can also reconsider a model for gonorrhea transmission [**2**, Section 4.6.1] and Macdonald's model for the transmission of schistosomiasis [**2**, Section 4.6.2] and rediscover [**2**, Theorems 4.8 and 4.9 and Theorems 4.12 and 4.13]. We also point out that, by an argument similar to that in Theorem 3.1 and 3.2, one can deduce the threshold theorem on the cooperative periodic systems of differential equations and autonomous and periodic reaction-diffusion systems with strict sublinearity, and hence improve [**2**, Theorem 4.5], [**2**, Theorems 5.1 and 5.5] and [**2**, Theorems 5.6 and 5.11], respectively. For some related results, we refer to [**21**].

Example 2. Consider the two-species competition chemostat model with delays (see [**5**] and [**12**]):

$$
\begin{aligned}
\frac{dS}{dt} &= (S^0 - S(t))D - \sum_{i=1}^{2} P_i(S(t))x_i(t) \\
\frac{dx_1}{dt} &= -Dx_1(t) + e^{-D\tau_1}P_1(S(t-\tau_1))x_1(t-\tau_1) \\
\frac{dx_2}{dt} &= -Dx_2(t) + e^{-D\tau_2}P_2(S(t-\tau_2))x_2(t-\tau_2)
\end{aligned}
\tag{4.2}
$$

where $D > 0$, $S^{(0)} > 0$, $\tau_i > 0$, $i = 1, 2$, and $P_i : [0, +\infty) \to [0, +\infty)$ are continuously differentiable functions satisfying $P_i(0) = 0$ and $P_i'(s) > 0$ for all $s \geq 0$. Let $z_1(t) = e^{D\tau_1}x_1(t+\tau_1), z_2(t) = e^{D\tau_2}x_2(t+\tau_2)$. Then it follows that

$$S(t) + z_1(t) + z_2(t) = S^{(0)} + O(e^{-Dt}), \quad \text{as } t \longrightarrow \infty,$$

and, hence, by asymptotically autonomous semiflow theory, some asymptotic behavior of systems (4.2) can be determined by the following limit system

$$
\begin{aligned}
\frac{dz_1}{dt} &= -Dz_1(t) + P_1(S^{(0)} - z_1(t) - z_2(t))e^{-D\tau_2}z_1(t-\tau_1) \\
\frac{dz_2}{dt} &= -Dz_2(t) + P_2(S^{(0)}) - z_1(t) - z_2(t))e^{-D\tau_2}z_2(t-\tau_2).
\end{aligned}
\tag{4.3}
$$

As usual, the first important discussion is on the scalar equation when one or other of the species is absent, that is, the case of just one species being grown alone in the chemostat. Here we then consider the following equation

$$(4.4)\quad \begin{cases} dz(t)/dt = P(S^{(0)} - z(t))e^{-D\tau}z(t-\tau) - Dz(t), & t > 0 \\ z_0 = \phi \in C([-\tau, 0], R) \quad \text{and} \quad 0 \le \phi(\theta) \le S^{(0)} & \\ & \text{for } \theta \in [-\tau, 0] \end{cases}$$

where $P \in C^1([0,\infty),[0,\infty))$ satisfies

$$(4.5)\qquad P(0) = 0 \quad \text{and} \quad P'(s) > 0 \quad \text{for all } s \ge 0.$$

Then we have the following threshold result. For a similar result, except for the stability of equilibria and a different approach, we refer to the main result [**5**, Theorem 3.4].

Proposition 4.2. *Let* $[\hat{0}, \hat{S}^{(0)}] = \{\phi \in C([-\tau, 0], R); 0 \le \phi(\theta) \le S^{(0)}, -\tau \le \theta \le 0\}$ *and assume that* (4.5) *holds.*

(a) *If* $P(S^{(0)}) \le De^{D\tau}$, *then* $z = \hat{0}$ *is globally asymptotically stable for* (4.4) *with respect to* $[\hat{0}, \hat{S}^{(0)}]$;

(b) *If* $P(S^{(0)}) > De^{D\tau}$, *then* (4.4) *admits a positive equilibrium* $\hat{z}^*$ *and* $z = \hat{z}^*$ *is globally asymptotically stable with respect to* $[\hat{0}, \hat{S}^{(0)}]\backslash\{\hat{0}\}$.

Proof. Let $\overline{P}(s)$, $s \in R$ be any continuously differentiable extension of $P(s)$ on $[0,\infty)$ to R satisfying $\overline{P}'(s) > 0$ for all $s \in R$. Consider then the following equation

$$(4.6)\qquad dz(t)/dt = f(z_t)$$

where $f : C([-\tau, 0], R) \to R$ is defined by

$$f(\phi) = \overline{P}(S^{(0)} - \phi(0))e^{-D\tau} \cdot \phi(-\tau) - D\phi(0).$$

Then $f \in C^1([-\tau, 0], R)$ and for any $\psi, \phi \in C([-\tau, 0], R)$,

$$\begin{aligned} df(\psi)\phi = &-\overline{P}'(S^{(0)} - \psi(0))\phi(0)e^{-D\tau}\psi(-\tau) \\ &+ \overline{P}(S^{(0)} - \psi(0))e^{-D\tau}\phi(-\tau) - D\phi(0) \end{aligned}$$

and hence for any $\psi \in [\hat{0}, \hat{S}^{(0)}]$ and any $\phi \in C^+([-\tau, 0], R)$ with $\phi(0) = 0$, $df(\psi)\phi = \overline{P}(S^{(0)} - \psi(0))e^{-D\tau}\phi(-\tau) \geq 0$, that is, $f : [\hat{0}, \hat{S}^{(0)}] \to R$ is cooperative on $[\hat{0}, \hat{S}^{(0)}]$. Since for any $\phi \in [\hat{0}, \hat{S}^{(0)}]$ with $\phi(0) = 0$ ($\phi(0) = S^{(0)}$), $f(\phi) = \overline{P}(S^{(0)})e^{-D\tau}\phi(-\tau) \geq 0$ ($f(\phi) = -D\phi(0) \leq 0$), by [**14**, Proposition 1.3], $[\hat{0}, \hat{S}^{(0)}]$ is positively invariant, that is, for any $\phi \in [\hat{0}, \hat{S}^{(0)}]$, the unique solution $z(t, \phi)$ of (4.6) with $z_0 = \phi$ satisfies $0 \leq z(t, \phi) \leq S^{(0)}$ on its maximal interval of existence $[0, \sigma_\phi)$. Moreover, for any $\phi \in V = [\hat{0}, \hat{S}^{(0)}]$ with $\phi \gg \hat{0}$, and any $\alpha \in (0, 1)$, since $\overline{P}'(s) > 0$ for all $s \geq 0$, $f(\alpha\phi) - \alpha f(\phi) = \alpha e^{-D\tau}\phi(-\tau)[\overline{P}(S^{(0)} - \alpha\phi(0)) - \overline{P}(S^{(0)} - \phi(0))] > 0$, that is, $f : V \to C([-\tau, 0], R)$ is strictly sublinear. Since $f(\hat{0}) = 0$ and for any $\phi \in C([-\tau, 0], R)$, $df(\hat{0})\phi = \overline{P}(S^{(0)})e^{-D\tau}\phi(-\tau) - D\phi(0)$, it is easy to see that $df(\hat{0})$ satisfies (R). Notice that $F(x) = f(\hat{x}) = \overline{P}(S^{(0)} - x)e^{-D\tau}x - Dx = x[\overline{P}(S^{(0)} - x)e^{-D\tau} - D]$ and $DF(0) = P(S^{(0)})e^{-D\tau} - D$, now Theorem 3.2 with Remark 2.3 completes the proof. □

Example 3. Consider the following systems of delay differential equations, which are deduced from a population model with dispersal and stage structure (see [**20**]).

$$\begin{aligned} \frac{dM_i(t)}{dt} &= -M_i(t)g_i(M_i(t)) + F_i(M_1(t), \dots, M_n(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij}\alpha_j M_j(t-\tau), \quad t \geq 0, \quad i = 1, 2, \dots, n, \end{aligned} \tag{4.7}$$

where $\tau > 0$, $\alpha_j > 0$, $b_{ij} \geq 0$, $g_i \in C^1(R^n_+, R)$, $F_i \in C^1(R^n_+, R)$ and $F_i(0) = 0$, $i, j = 1, 2, \dots, n$. By a careful verification of all assumptions in Theorem 3.2, we have the following threshold result.

Proposition 4.3. *Assume that*

(1) $F = (F_1, F_2, \dots, F_n)^T : R^n_+ \to R^n$ *is cooperative and irreducible, i.e., for any* $M \in R^n_+$, $\partial F_i/\partial M_j \geq 0$, $i \neq j$, *and* $DF(M) = (\partial F_i/\partial M_j)_{1 \leq i,j \leq n}$ *is irreducible;*

(2) *for any* $M \in R^n_+$ *with* $M_i = 0$, $F_i(M) \geq 0$, $i = 1, 2, \dots, n$, *and for any* $1 \leq j \leq n$, $B_j = (b_{1j}, b_{2j}, \dots, b_{nj})^T > 0$ *in* R^n;

(3) $H = (H_1, H_2, \dots, H_n)^T : R^n_+ \to R^n$ *is strictly sublinear, i.e., for any* $M \gg 0$ *and any* $\alpha \in (0, 1)$, $H(\alpha M) > \alpha H(M)$, *where*

$H_i(M) = -M_i g_i(M_i) + F_i(M)$, $i = 1, 2, \ldots, n$. Let $A = (A_{ij})_{1\le i,j\le n}$ be defined by $A_{ij} = -g_i(0) + \partial F_i(0)/\partial M_j + b_{ij}\alpha_j$, $1 \le i$, $j \le n$.

(a) *If $s(A) = \max\{\mathrm{Re}\,\lambda; \det(\lambda I - A) = 0\} \le 0$, then $\hat{0}$ is globally asymptotically stable for* (4.7) *with respect to $C^+ = C^+([-\tau, 0], R^n)$;*

(b) *If $s(A) > 0$, then either*

(i) *for any $\phi \in C^+\backslash\{\hat{0}\}$, $\lim_{t\to\infty} \|M_t(\phi)\| = +\infty$, where $M(t,\phi)$ is the unique solution of* (4.7) *with $M_0 = \phi$; or alternatively,*

(ii) (4.7) *admits a unique positive equilibrium $\hat{M}^*$ with $M^* \gg 0$ in R^n and $\hat{M}^*$ is globally asymptotically stable with respect to $C^+\backslash\{\hat{0}\}$.*

By [**20**, Lemma 3.1] it follows that if $\liminf_{x\to\infty} g_i(x) > \sum_{j=1}^n b_{ij}\alpha_j$, $1 \le i \le n$, and for each $1 \le i \le n$, $F_i(M) \le 0$ for any $M \in R^n_+$ with $M_i \ge M_j$, $1 \le j \le n$, then system (4.7) is point dissipative, i.e., solutions of (4.7) are ultimately bounded. It also easily follows that if $g_i(0) < \sum_{j=1}^n b_{ij}\alpha_j$ then $s(A) > 0$. Therefore, the alternative (b) in Proposition 4.3 implies [**20**, Theorem 2.1].

Remark 4.3. In [**13**], Z. Lu and Y. Takeuchi discussed the single-species discrete diffusion systems

$$\frac{dx_i}{dt} = x_i f_i(x_i) + \sum_{j=1, j\ne i}^{n} D_{ij}(x_j - \alpha_{ij}x_i) = F_i(x), \quad 1 \le i \le n, \tag{4.8}$$

where $D_{ij} \ge 0$, $\alpha_{ij} \ge 0$, $1 \le i, j \le n$, and $f_i : R_+ \to R$. It is easy to see that our Corollaries 3.1 and 3.2 imply their main results (see [**13**, Theorems 1 and 2]). In particular, we point out that the key assumption in [**13**, Theorem 2], i.e.,

(H) $f_i(0) > 0$, $f_i'(x_i) < 0$ for $x_i > 0$, $i = 1, 2, \ldots, n$

is just a sufficient condition for the strict sublinearity of $F = (F_1, \ldots, F_n)^T$: $R^n_+ \to R^n$.

REFERENCES

1. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

2. V. Capasso, *Mathematical structures of epidemic systems*, Springer-Verlag, Berlin, 1993.

3. E.N. Dancer, *Fixed point index calculations and applications*, in *Topological nonlinear analysis, degree, singularity and variation* (M. Matzeu and A. Vignili, eds.), Birkhauser, Boston, 1995.

4. E.N. Dancer and P. Hess, *Stability of fixed points for order-preserving discrete-time dynamical systems*, J. Reine Angew. Math. **419** (1991), 125–139.

5. S.F. Ellermeyer, *Competition in the chemostat: Global asymptotic behavior of a model with delayed response in growth*, SIAM J. Appl. Math. **54** (1994), 456–465.

6. J.K. Hale, *Theory of functional differential equations*, Springer-Verlag, Berlin, 1977.

7. ———, *Asymptotic behavior of dissipative systems*, Math. Surveys Monographs **25**, Amer. Math. Soc., Providence, R.I., 1988.

8. P. Hess, *Periodic-parabolic boundary value problems and positivity*, Pitman Res. Notes Math. Ser. **247**, Longman Sci. Tech., Harlow, 1991.

9. M.W. Hirsch, *The dynamical systems approach to differential equations*, Bull. Amer. Math. Soc. **11** (1984), 1–64.

10. ———, *Stability and convergence in strongly monotone dynamical systems*, J. Reine Angew. Math. **383** (1988), 1–53.

11. ———, *Positive equilibria and convergence in subhomogeneous monotone dynamics*, in *Comparison methods and stability theory* (X. Liu and D. Siegel, eds.), Lecture Notes in Pure and Applied Math **162**, Marcel Dekker, New York, 1994.

12. S.-B. Hsu, P. Waltman and S.F. Ellermeyer, *A remark on the global asymptotic stability of a dynamical system modeling two species competition*, Hiroshima Math. J. **24** (1994), 435–445.

13. Z. Lu and Y. Takeuchi, *Global asymptotic behavior in single-species discrete diffusion systems*, J. Math. Biol. **32** (1993), 67–77.

14. R.H. Martin and H.L. Smith, *Reaction-diffusion systems with time delays: Monotonicity, invariance, comparison and convergence*, J. Reine Angew. Math. **413** (1991), 1–35.

15. H.L. Smith, *Cooperative systems of differential equations with concave nonlinearities*, Nonlinear Anal. **10** (1986), 1037–1052.

16. ———, *Invariant curves for mappings*, SIAM J. Math. Anal. **17** (1986), 1053–1067.

17. ———, *Monotone semiflows generated by functional differential equations*, J. Differential Equations **66** (1987), 420–442.

18. ———, *Systems of ordinary differential equations which generate an order preserving flow, A survey of results*, SIAM Rev. **30** (1988), 87–113.

19. P. Takáč, *Asymptotic behavior of discrete-time semigroups of sublinear, strongly increasing mappings with applications in biology*, Nonlinear Anal. **14** (1990), 35–42.

20. J.-H. Wu and H.I. Freedman, *Global stability in a population model with dispersal and stage structure*, in *Differential equations models in biology, epidemiology and ecology* (S. Busenberg and M. Martelli, eds.), Lecture Notes in Biomath. **92** (1991), 257–267.

21. X.-Q. Zhao, *Global attractivity and stability in some monotone discrete dynamical systems*, Bull. Austral. Math. Soc. **53** (1996), 305–324.

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