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Permanence Implies the Existence of Interior Periodic Solutions for FDEs

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Dedicated to Professor Zhicheng Wang on the occation of his retirement.

Abstract. By appealing to the theory of global attractors and steady states for uniformly persistent dynamical systems, we show that uniform persistence implies the existence of interior periodic solutions for dissipative periodic functional differential equations of retarded and neutral type. This result is then applied to a multi-species competitive system and a SIS epidemic model.

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1. Introduction

Periodic solutions of periodic functional differential equations (FDEs) have received extensive investigations, see, e.g., [3, 5, 14, 15, 11] and references therein. It is well-known that the existence, uniqueness, stability, and attractivity of periodic solutions for a periodic semiflow are equivalent to those of fixed points for its Poincaré map(see, e.g., [17]). Various fixed point theorems, coincidence degree theory, and bifurcation methods can be used to study the existence of positive periodic solutions for periodic FDEs, while the uniqueness and global attractivity of these positive solutions may be addressed via the monotone operators approach and Liapunov function techniques.

Uniform persistence is an important concept in population dynamics, since it characterizes the long-term survival of some or all interacting species in an ecosystem. Looked at abstractly, it is the notion that a closed subset of the state space is repelling for the dynamics on the complementary set, and then it gives a uniform estimate

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for omega limit sets, which sometimes is essential to obtain a more detailed global dynamics. For the general theory of uniform persistence in dissipative systems, we refer to [4, 12, 17] and references therein.

A dissipative and uniformly persistent system is often said to be permanent. In [16, 6], it was shown that permanence implies the existence of at least one coexistence steady state for abstract discrete- and continuous-time dynamical systems under appropriate assumptions. This result provides a dynamic approach to some static problems, e.g., the existence of positive steady states and periodic solutions for evolution equations. We should point out that this approach is very flexible in its applications since the concept of uniform persistence depends on the choice of an open subset in the phase space. The purpose of this paper is to show that permanence implies the existence of interior periodic solutions for periodic functional differential equations by appealing to an abstract result in [6].

The remaining parts of this paper are organized as follows. In the next section, we present the necessary preliminary concepts and results on discrete-time dynamical systems. In section 3, we establish the existence of interior periodic solutions for three types of periodic functional differential equations. Section 4 is aimed at the applications of the general results to two periodic and time-delayed systems in population biology. A threshold dynamics is also obtained for the SIS epidemic model.

2. Preliminaries

Let (M, d) be a complete metric space. Recall that a set U in M is said to be a neighborhood of another set V provided V is contained in the interior $int(U)$ of U. For any subsets A, $B \subset M$, we define

$$d(x, A) := \inf_{y \in A} d(x, y), \quad \delta(B, A) := \sup_{x \in B} d(x, A).$$

The *Kuratowski measure of noncompactness*, α, is defined by

$$\alpha(B) = \inf\{r : B \text{ has a finite open cover of diameter } \leq r\},$$

for any bounded set B of M. It is easy to see that B is precompact (i.e., $\overline{B}$ is compact) if and only if $\alpha(B) = 0$.

Let $T : M \to M$ be a continuous map. We consider the discrete-time dynamical system $T^n : M \to M$, $\forall n \geq 0$, which is defined by

$$T^0 = I, \quad T^n = T \circ T^{n-1}, \quad \forall n \geq 1.$$

For a subset $C \subset M$, we denote its positive orbit for T as $\gamma^+(C) := \bigcup_{n \geq 0} T^n(C)$. A subset $A \subset M$ is *positively invariant* for T if $T(A) \subset A$. A is *invariant* for T if $T(A) = A$. We say that a subset $A \subset M$ *attracts* a subset $B \subset M$ for T if $\lim_{n \to \infty} \delta(T^n(B), A) = 0$.

Definition 2.1. *A continuous mapping $T : M \to M$ is said to be point (compact) dissipative if there is a bounded set B_0 in M such that B_0 attracts each point (compact set) in M; T is compact if T maps any bounded set in M to a precompact set; T is α-condensing (α-contraction of order k, $0 \le k < 1$) if T takes bounded sets to bounded sets and $\alpha(T(B)) < \alpha(B)$ ($\alpha(T(B)) \le k\alpha(B)$) for any nonempty closed bounded set $B \subset M$ with $\alpha(B) > 0$.*

Clearly, a compact map is an α-contraction of order 0, and an α-contraction of order k is α-condensing.

Definition 2.2. *A nonempty, compact and invariant set $A \subset M$ is said to be an attractor for T if A attracts one of its neighborhoods; a global attractor for T if A is an attractor that attracts every point in M; a strong global attractor for T if A attracts every bounded subset of M.*

For the general theory of global attractors in dissipative systems, we refer to [2, 8, 6] and references therein.

Let M_0 be a nonempty open subset of M. We define $\partial M_0 := M \setminus M_0$. Clearly, ∂M_0 is a closed subset of M.

Definition 2.3. *Let $T : M \to M$ be a continuous map with $T(M_0) \subset M_0$. T is said to be uniformly persistent with respect to M_0 if there exists $\eta > 0$ such that $\liminf_{n\to+\infty} d\left(T^n(x), \partial M_0\right) \ge \eta$ for all $x \in M_0$.*

In what follows, we assume that M is a closed subset of a Banach space $(X, \|\cdot\|)$, that M_0 is a convex and relatively open subset in M, and that $T : M \to M$ is a continuous map with $T(M_0) \subset M_0$.

The following result on the existence of the global attractor and fixed point of T in M_0 comes from [6, Theorem 4.5] with $\rho(x) = d(x, \partial M_0)$, which is a generalization of [16, Theorem 2.3].

Theorem 2.1. *Assume that*

(1) T is point dissipative and uniformly persistent.

(2) One of the following two conditions hold:

  (2a) T^{n_0} is compact for some integer $n_0 \ge 1$, or

  (2b) Positive orbits of compact sets in M are bounded.

(3) T is α-condensing.

Then $T : M_0 \to M_0$ admits a global attractor A_0, and T has a fixed point in A_0.

3. Periodic FDEs

In this section, we establish the existence of periodic solutions for three types of periodic functional differential equations by applying Theorem 2.1 to their time period maps.

For a given $\tau \geq 0$, let $C := C([-\tau, 0], \mathbb{R}^m)$. For any function $x : [-\tau, b) \to \mathbb{R}^m$, $b > 0$, we define $x_t \in C$ by $x_t(\theta) = x(t+\theta)$, $\forall \theta \in [-\tau, 0]$.

We first consider a periodic retarded functional differential equation

$$\frac{dx(t)}{dt} = f(t, x_t) \tag{3.1}$$

with $f : \mathbb{R} \times C \to \mathbb{R}^m$ a continuous and bounded map, $f(t, \phi)$ a locally Lipschitz function in ϕ, and $f(t+\omega, \phi) = f(t, \phi)$ for some $\omega > 0$.

Theorem 3.1. *Let M be a closed subset of C and M_0 be a convex and relatively open subset in M such that for any $\phi \in M$, the unique solution $x(t, \phi)$ of (3.1) with $x_0 = \phi$ exists on $[0, \infty)$, and both M and M_0 are positively invariant for solution maps x_t, $t \geq 0$. Assume that*

(i) Solutions of (3.1) are ultimately bounded in M in the sense that there is a number $K_0 > 0$ such that $\limsup_{t\to\infty} \|x_t(\phi)\| \leq K_0$ for all $\phi \in M$.

(ii) Solutions of (3.1) are uniformly persistent with respect to M_0 in the sense that there exists $\eta > 0$ such that $\liminf_{t\to+\infty} d(x_t(\phi), \partial M_0) \geq \eta$ for all $\phi \in M_0$.

Then system (3.1) has at least one ω-periodic solution $x^*(t)$ with $x^*_t \in M_0$ for all $t \geq 0$.

Proof. By the proof of [2, Theorem 4.1.1], it follows that the solution map x_t is an α-contraction on M for an equivalent norm in C for each $t > 0$, and is compact on M for each $t \geq \tau$. Let T be the Poincaré map associated with system (3.1), that is, $T\phi = x_\omega(\phi)$, $\forall \phi \in M$. Clearly, there exists an integer $n_0 \geq 1$ such that $n_0\omega \geq \tau$. Thus, condition (2a) holds for T on M. Now Theorem 2.1 completes the proof. $\square$

In the case where $\tau = 0$, (3.1) reduces to a periodic ordinary differential system $x' = f(t, x)$, and hence, Theorem 3.1 also applies to periodic ODEs.

In Sections 8.4-8.6 of [5], sufficient conditions for permanence were obtained for three classes of nonautonomous and time-delayed Kolmogorov-type population models, respectively. If, in addition, such a system is ω-periodic, then Theorem 3.1 implies that it has at least one positive ω-periodic solution.

Suppose that $A(t, \theta)$ is a continuous $m \times m$ matrix function and $D_0 : \mathbb{R} \to L(C, \mathbb{R}^m)$ is a C^1-continuous map such that both A and D_0 are ω-periodic in t and the zero solution of the linear functional equation

$$D_0(t)y_t = 0, \quad y_0 = \phi$$

is uniformly asymptotically stable for $\phi \in C_{D_0} =: \{\phi \in C : D_0(0)\phi = 0\}$. For a given $t \in \mathbb{R}$, we define $D(t) : C \to \mathbb{R}^m$ by

$$D(t)\phi = D_0(t)\phi + \int_{-\tau}^{0} A(t,\theta)\phi(\theta)d\theta.$$

Next we consider a periodic neutral function differential equation

$$\frac{d}{dt}D(t)x_t = f(t, x_t) \tag{3.2}$$

with $f(t,\phi)$ defined as in system (3.1).

Theorem 3.2. *Let M be a closed subset of C and M_0 be a convex and relatively open subset in M such that for any $\phi \in M$, the unique solution $x(t,\phi)$ of (3.2) with $x_0 = \phi$ exists on $[0,\infty)$, and both M and M_0 are positively invariant for solution maps x_t, $t \geq 0$. Assume that*

(i) Solutions of (3.2) are ultimately bounded in M.

(ii) Solutions of (3.2) are uniformly persistent with respect to M_0.

Then system (3.2) has at least one ω-periodic solution $x^*(t)$ with $x^*_t \in M_0$ for all $t \geq 0$.

Proof. It was shown in [7] that point dissipative implies compact dissipative for solution maps x_t of (3.2)(see also [2, Theorem 4.5.6]). Let T be the Poincaré map associated with the system (3.2). Then condition (2b) holds for T on M. By [2, Section 4.5.3]), it follows that T is an α-contraction on M for an equivalent norm in C. Now Theorem 2.1 completes the proof. □

In the case where $D_0(t)\phi = \phi(0)$ and $A(t,\theta) \equiv 0$, the neutral equation (3.2) reduces to the retarded equation (3.1), and hence, Theorem 3.1 is also a consequence of Theorem 3.2.

For a given $\gamma > 0$, we define the phase space as

$$C_\gamma = \{\phi \in C((-\infty,0],\mathbb{R}^m) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\}$$

equipped with the norm $\|\phi\| = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|\phi(\theta)|$.

Finally, we consider a retarded functional differential equation with infinite delay

$$\frac{dx(t)}{dt} = f(t, x_t) \tag{3.3}$$

with $f : \mathbb{R} \times C_\gamma \to \mathbb{R}^m$ a continuous and bounded map, $f(t,\phi)$ a locally Lipschitz function in ϕ, and $f(t+\omega,\phi) = f(t,\phi)$ for some $\omega > 0$.

Theorem 3.3. *Let M be a closed subset of C_γ and M_0 be a convex and relatively open subset in M such that for any $\phi \in M$, the unique solution $x(t,\phi)$ of (3.3) with $x_0 = \phi$ exists on $[0,\infty)$, and both M and M_0 are positively invariant for solution maps x_t, $t \geq 0$. Assume that*

(i) Solutions of (3.3) are ultimately bounded in M.

(ii) Solutions of (3.3) are uniformly persistent with respect to M_0.

(iii) Solutions of (3.3) are uniformly bounded in M in the sense that for any $L > 0$, there exists $K = K(L) > 0$ such that $\|x_t(\phi)\| \leq K$ for all $t \geq 0$ and $\phi \in M$ with $\|\phi\| \leq L$.

Then system (3.3) has at least one ω-periodic solution $x^*(t)$ with $x_t^* \in M_0$ for all $t \geq 0$.

Proof. Let T be the Poincaré map associated with the system (3.3). Clearly, the assumption (iii) implies that positive orbits of bounded sets in M for T are bounded. By the arguments in [2, Section 4.9.1], it then follows that T is an α-contraction on M for an equivalent norm in C_γ. Now Theorem 2.1 completes the proof. □

Compared with Theorems 3.1 and 3.2, we may wonder whether the assumption (iii) can be removed from Theorem 3.3. This remains an open problem for system (3.3).

4. Applications

In applications of the results in section 3, the key point is to prove a given time-delayed system is uniformly persistent for an appropriate open set M_0. In this section, we provide two illustrative examples.

Let the Banach space C be defined as in section 3, and C^+ be its positive cone consisting of all nonnegative functions in C. We consider m-species periodic competitive system with time delay

$$\frac{dx_i(t)}{dt} = x_i(t)\left(g_i(t, x_i(t)) - h_i(t, x_t)\right), \; 1 \leq i \leq m, \tag{4.1}$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. We assume that

(A1) For each i, $g_i \in C^1(\mathbb{R}^2, \mathbb{R})$ and $h_i \in C^1(\mathbb{R}\times C, \mathbb{R})$ are ω-periodic in t, $h_i(t,\phi) \geq 0$ for all $(t,\phi) \in [0,\omega] \times C^+$, $h_i(t,\phi)$ is nondecreasing in $\phi \in C^+$, and $\frac{\partial g_i(t,x_i)}{\partial x_i} < 0$ for all $(t, x_i) \in [0,\omega] \times \mathbb{R}_+$.

(A2) For each i, there exists $K_i > 0$ such that $g_i(t, K_i) \leq 0$ for all $t \in [0,\omega]$, and $\int_0^\omega g_i(t,0)dt > 0$.

By [17, Theorem 5.2.1], it follows that the periodic ordinary differential equation

$$x_i'(t) = x_i(t)g_i(t, x_i(t))$$

has a unique positive ω-periodic solution $x_i^*(t)$, which is globally attractive in $\mathbb{R}_+ \setminus \{0\}$. We further assume that

(A3) $\int_0^\omega g_i(t, 0)dt > \int_0^\omega h_i(t, x_t^*)dt$ for each $1 \le i \le m$.

Then we have the following result.

Theorem 4.1. *Let assumptions (A1)-(A3) hold. Then system (4.1) is uniformly persistent and has at least one positive ω-periodic solution.*

Proof. For a given $\phi \in C^+$, let $x(t, \phi) = (x_1(t), \ldots, x_m(t))$ be the unique solution of system (4.1) with $x_0 = \phi$. Then we have

$$x_i'(t) \le x_i(t)g_i(t, x_i(t)).$$

By the comparison theorem, it follows that $x(t, \phi)$ exists for all $t \ge 0$, and for any $\epsilon > 0$ there exists $t_1 = t_1(\phi) > 0$ such that

$$0 \le x_i(t) \le x_i^*(t) + \epsilon, \ \forall t \ge t_1, \ 1 \le i \le m. \tag{4.2}$$

Define

$$M := C^+, \quad M_0 = \{\phi = (\phi_1, \ldots, \phi_m) \in M : \phi_i(0) > 0, \forall 1 \le i \le m\}.$$

It is easy to see that both M and M_0 are positively invariant for solution maps of (4.1). Now we assume that $\phi \in M_0$, and let $\bar{\epsilon} := (\epsilon, \ldots, \epsilon) \in \mathbb{R}^m$. Then

$$x_i'(t) \ge x_i(t) \left(g_i(t, x_i(t)) - h_i(t, x_t^* + \bar{\epsilon}) \right), \ \forall t \ge t_1 + \tau.$$

We fix a sufficiently small number ϵ such that

$$\int_0^\omega g_i(t, 0)dt > \int_0^\omega h_i(t, x_t^* + \bar{\epsilon})dt, \ \forall 1 \le i \le m.$$

It follows from [17, Theorem 5.2.1] that each periodic ordinary differential equation

$$y_i'(t) = y_i(t) \left(g_i(t, y_i(t)) - h_i(t, x_t^* + \bar{\epsilon}) \right)$$

has a globally attractive positive ω-periodic solution $y_i^*(t)$ in $\mathbb{R}_+ \setminus \{0\}$. Choose a number $\beta > 0$ such that $\beta < \min_{t \in [0,\omega]} y_i^*(t)$ for all $1 \le i \le m$. Note that $x_i(t) > 0$ for all $t \ge 0$. Thus, the comparison theorem implies that there exists $t_2 = t_2(\phi) \ge t_1 + \tau$ such that

$$x_i(t) \ge y_i^*(t) - \beta, \ \forall t \ge t_2, \ 1 \le i \le m. \tag{4.3}$$

By (4.2) and (4.3), we see that solutions of system (4.1) are ultimately bounded in M and uniformly persistent with respect to M_0. Now Theorem 3.1 completes the proof. $\square$

Consider a time-delayed SIS epidemic model, which was presented in [1]:

$$\begin{aligned} S'(t) &= B(N(t-\tau))N(t-\tau)e^{-d_1\tau} - dS(t) - \frac{\beta S(t)I(t)}{N(t)} + \gamma I(t) \\ I'(t) &= \frac{\beta S(t)I(t)}{N(t)} - (d + d_2 + \gamma)I(t) \end{aligned} \tag{4.4}$$

with $N(t) = S(t) + I(t)$. Here $d > 0$ is the death rate constant at the adult stage, and $B(N)$ is a birth rate function, τ is the developmental or maturation time, $d_1 > 0$ is the death rate constant for the life stage prior to the adult stage, $d_2 \geq 0$ is the disease induced death rate constant, $\gamma > 0$ is the recovery rate constant ($\frac{1}{\gamma}$ is the average infective time), and $\beta > 0$ is the contact rate constant. The standard incidence function is used with $\frac{\beta I}{N}$ giving the average number of adequate contacts with infectives of one susceptible per unit time.

Now we assume the contact rate is a nonnegative continuous ω-periodic function $\beta(t)$ with its time average over $[0, \omega]$ satisfying

$$\bar{\beta} := \frac{1}{\omega} \int_0^\omega \beta(t)dt > 0.$$

For simplicity, we take the birth rate function as $B(N) = \frac{a}{N} + c$ with $a > 0$ and $0 < c < d$. Then we obtain the following periodic and time-delayed SIS epidemic model

$$\begin{aligned} S'(t) &= (a + cN(t-\tau))\, e^{-d_1\tau} - dS(t) - \frac{\beta(t)S(t)I(t)}{N(t)} + \gamma I(t) \\ I'(t) &= \frac{\beta(t)S(t)I(t)}{N(t)} - (d + d_2 + \gamma)I(t). \end{aligned} \tag{4.5}$$

As noticed in [18], the function $\frac{SI}{S+I}$ can be extended to a globally Lipschitz function $F(S, I)$ on $\mathbb{R}_+^2$ with $F(0, 0) = 0$. In what follows, we always identify the term $\frac{S(t)I(t)}{N(t)}$ with $F(S(t), I(t))$. By [9, Theorem 5.2.1], for any $\phi \in M := C([-\tau, 0], \mathbb{R}_+^2)$, there is a unique solution $(S(t, \phi)), I(t, \phi))$ of system (4.5) with $(S(\theta, \phi), I(\theta, \phi)) = \phi(\theta), \forall \theta \in [-\tau, 0]$, and $S(t, \phi) \geq 0$, $I(t, \phi) \geq 0$ for all $t \geq 0$ in its maximal interval of existence. Then $N(t) = S(t) + I(t)$ satisfies the differential inequality

$$N'(t) \leq (a + cN(t-\tau))e^{-d_1\tau} - dN(t).$$

Since $0 < c < d$, it is easy to see that the linear equation

$$N'(t) = (a + cN(t-\tau))e^{-d_1\tau} - dN(t) \tag{4.6}$$

has a unique positive equilibrium $N_e := \dfrac{a}{d - ce^{-d_1\tau}}$, which is globally stable in $C([-\tau, 0], \mathbb{R}_+)$ (see, e.g., [1, Theorem 3.1]). By the comparison theorem (see [9, Theorem 5.1.1]), it then follows that each solution $(S(t, \phi), I(t, \phi))$ exists globally on $[0, \infty)$, and solutions of system (4.5) are ultimately bounded in M. Moreover, for any

$\phi = (\phi_1, \phi_2) \in M$ with $\phi_2(0) > 0$, by using two equations in (4.5) respectively, we have $S(t, \phi) > 0, I(t, \phi) > 0$, $\forall t > 0$.

Define

$$M_0 := \{\phi = (\phi_1, \phi_2) \in M : \phi_2(0) > 0\}, \quad \partial M_0 := M \setminus M_0.$$

Then we have the following threshold type result.

Theorem 4.2. *Let $R_0 := \frac{\bar{\beta}}{d+d_2+\gamma}$. Then the following statement is valid:*

(i) If $R_0 < 1$, then any solution $(S(t, \phi), I(t, \phi))$ of system (4.5) with $\phi \in M_0$ satisfies $\lim_{t\to\infty} S(t, \phi) = N_e$ and $\lim_{t\to\infty} I(t, \phi) = 0$;

(ii) If $R_0 > 1$, system (4.5) has a positive ω-periodic solution, and there is $\eta > 0$ such that any solution $(S(t, \phi), I(t, \phi))$ of system (4.5) with $\phi \in M_0$ satisfies $\liminf_{t\to\infty} S(t, \phi) \geq \eta$ and $\liminf_{t\to\infty} I(t, \phi) \geq \eta$.

Proof. Define $\Phi(t)\phi = (S_t(\phi), I_t(\phi))$, $t \geq 0$, $\phi \in M$. Then $\Phi(t)$ is an ω-periodic semiflow on M. Fix a sufficiently small positive number δ_0 such that

$$B(N) = \frac{a}{N} + c > 2(d + d_2)e^{d_1\tau}, \quad \forall N \in (0, 2\delta_0].$$

Since $\lim_{\phi\to 0} \Phi(t)\phi = 0$ uniformly for $t \in [0, \omega]$, there exists $\delta_1 > 0$ such that

$$\|\Phi(t)\phi\| \leq \delta_0, \quad \forall t \in [0, \omega], \ \|\phi\| \leq \delta_1.$$

Claim 1. $\limsup_{n\to\infty} \|\Phi(n\omega)\phi\| \geq \delta_1$ for all $\phi \in M_0$.

Suppose that, by contradiction, $\limsup_{n\to\infty} \|\Phi(n\omega)\phi\| < \delta_1$ for some $\phi \in M_0$. Then there exists an integer $N_1 \geq 1$ such that $\|\Phi(n\omega)\phi\| < \delta_1$, $\forall n \geq N_1$. Let $(S(t), I(t)) = (S(t, \phi), I(t, \phi))$, $t \geq 0$. For any $t \geq N_1\omega$, we have $t = n\omega + t'$ with $n \geq N_1$ and $t' \in [0, \omega]$. Thus, we have

$$\|\Phi(t)\phi\| = \|\Phi(t')(\Phi(n\omega)\phi)\| \leq \delta_0, \quad \forall t \geq N_1\omega.$$

Since $N(t) \geq I(t) > 0, \forall t \geq 0$, $N(t)$ satisfies the following differential inequality

$$N'(t) \geq 2(d + d_2)N(t - \tau) - (d + d_2)N(t), \quad \forall t \geq t_1 := N_1\omega + \tau.$$

By [9, Corollary 5.5.2], the stability modulus, denoted by s, of the linear time-delayed equation

$$N'(t) = 2(d + d_2)N(t - \tau) - (d + d_2)N(t) \tag{4.7}$$

is positive. Thus, [9, Theorem 5.5.1] implies that (4.7) admits a solution $N^*(t) = e^{st}u$ with $u > 0$. Since $N(t) > 0, \forall t > 0$, we can choose a small number $k > 0$ such that $N(t) > kN^*(t), \forall t \in [t_1 - \tau, t_1]$. By the comparison theorem (see [9, Theorem 5.1.1]), we then obtain $N(t) \geq kN^*(t)$, $\forall t \geq t_1$. This implies $\lim_{t\to\infty} N(t) = \infty$, a contradiction.

In the case where $R_0 < 1$, we have $\bar{\beta} < d + d_2 + \gamma$. If $I(0) > 0$, then $N(t) \geq I(t) > 0, \forall t \geq 0$, and hence, by equation (4.5), we obtain

$$I'(t) \leq (\beta(t) - (d + d_2 + \gamma))I(t), \quad \forall t \geq 0.$$

It follows that

$$I(t) \leq I(0)e^{\int_0^t (\beta(s)-(d+d_2+\gamma))ds}, \quad \forall t \geq 0.$$

Since $\frac{1}{\omega}\int_0^{\omega}(\beta(t) - (d + d_2 + \gamma))dt = \bar{\beta} - (d + d_2 + \gamma) < 0$, we see that $\lim_{t\to\infty} I(t) = 0$. Thus, $N(t)$ satisfies the following nonautonomous time-delayed equation

$$N'(t) = (a + cN(t - \tau))e^{-d_1 T} - dN - d_2 I(t)$$

which is asymptotic to the autonomous time-delayed equation (4.6). By [13, Theorem 4.1], together with Claim 1 above, it then follows that $\lim_{t\to\infty} N(t) = N_e$, and hence, $\lim_{t\to\infty} S(t, \phi) = \lim_{t\to\infty}(N(t) - I(t)) = N_e$.

In the case where $R_0 > 1$, we have $\bar{\beta} > d + d_2 + \gamma$. We consider the Poincaré map $T := \Phi(\omega) : M \to M$. Thus, $T^n\phi = \Phi(n\omega)\phi$, $\forall n \geq 0$, $\phi \in M$. Let $M_1 = (0, 0)$ and $M_2 = (N_e, 0)$. Then we have $\tilde{A}_\partial := \cup_{\phi\in\partial M_0}\omega(\phi) = \{M_1, M_2\}$, where $\omega(\phi)$ is the omega limit set of ϕ for the map T. Clearly, M_1 and M_2 are disjoint, compact and isolated invariant sets for T in ∂M_0, and no subset of $\{M_1, M_2\}$ forms a cycle in ∂M_0. Fix a number $\eta_0 \in \left(\frac{d+d_2+\gamma}{\beta}, 1\right)$. Since $\lim_{(I,N)\to(0,N_e)} \dfrac{N - I}{N} = 1 > \eta_0$, there exists $\eta_1 > 0$ such that

$$\frac{N - I}{N} > \eta_0, \quad \forall I \in [0, \eta_1],\ |N - N_e| \leq 2\eta_1.$$

Since $\lim_{\phi\to(N_e,0)} \Phi(t)\phi = (N_e, 0)$ uniformly for $t \in [0, \omega]$, there exists $\eta_2 > 0$ such that

$$\|\Phi(t)\phi - (N_e, 0)\| \leq \eta_1, \quad \forall t \in [0, \omega],\ \|\phi - (N_e, 0)\| \leq \eta_2.$$

Claim 2. $\limsup_{n\to\infty} \|\Phi(n\omega)\phi - (N_e, 0)\| \geq \eta_2$ for all $\phi \in M_0$.

Suppose that, by contradiction, $\limsup_{n\to\infty} \|\Phi(n\omega)\phi-(N_e, 0)\| < \eta_2$ for some $\phi \in M_0$. Then there exists an integer $N_2 \geq 1$ such that $\|\Phi(n\omega)\phi - (N_e, 0)\| < \eta_2, \forall n \geq N_2$. For any $t \geq N_2\omega$, we have $t = n\omega + t'$ with $n \geq N_2$ and $t' \in [0, \omega]$. Thus, we have

$$\|\Phi(t)\phi - (N_e, 0)\| = \|\Phi(t')(\Phi(n\omega)\phi) - (N_e, 0)\| \leq \eta_1, \quad \forall t \geq N_2\omega.$$

Thus, $I(t)$ satisfies the following differential inequality

$$I'(t) \geq (\beta(t)\eta_0 - (d + d_2 + \gamma))\, I(t), \quad \forall t \geq N_2\omega.$$

By the comparison theorem, it follows that

$$I(t) \geq I(N_2\omega)e^{\int_{N_2\omega}^t (\beta(s)\eta_0-(d+d_2+\gamma))ds}, \quad \forall t \geq N_2\omega.$$

Since

$$\frac{1}{\omega}\int_0^{\omega}(\beta(t)\eta_0-(d+d_2+\gamma))\,dt=\bar{\beta}\eta_0-(d+d_2+\gamma)>0,$$

we have $\lim_{n\to\infty} I(n\omega)=\infty$, a contradiction.

In view of two claims above, we see that M_1 and M_2 are isolated invariant sets for T in M, and $W^s(M_i)\cap M_0=\emptyset$, $i=1,2$, where $W^s(M_i)$ is the stable set of M_i for T. By the acyclicity theorem on uniform persistence for maps (see [17, Theorem 1.3.1 and Remark 1.3.1]), it follows that $T: M\to M$ is uniformly persistent with respect to M_0. Thus, [17, Theorem 3.1.1] implies that the periodic semiflow $\Phi(t): M\to M$ is also uniformly persistent with respect to M_0. By Theorem 3.1, we see that system (4.5) has an ω-periodic solution $(S^*(t), I^*(t))$ with $(S_t^*, I_t^*)\in M_0$ for all $t\geq 0$. Clearly, $S^*(t)>0$ and $I^*(t)>0$ for all $t>0$.

It remains to prove the practical uniform persistence. By Theorem 2.1, $T: M_0\to M_0$ has a global attractor A_0. Since $A_0=TA_0=\Phi(\omega)A_0$, it follows that $\phi_1(0)>0$ and $\phi_2(0)>0$ for all $\phi\in A_0$. Let $C:=\cup_{t\in[0,\omega]}\Phi(t)A_0$. It is easy to see that $C\subset M_0$ and $\lim_{t\to\infty} d(\Phi(t)\phi, C)=0$ for all $\phi\in M_0$. Define a continuous function $p: M\to\mathbb{R}_+$ by

$$p(\phi)=\min(\phi_1(0),\phi_2(0)),\quad \forall\phi=(\phi_1,\phi_2)\in M.$$

Clearly, $p(\phi)>0$, $\forall\phi\in C$. Since C is a compact subset of M_0, we have $\min_{\phi\in C} p(\phi)>0$. Consequently, there exists $\eta>0$ such that

$$\liminf_{t\to\infty}\min\left(S(t,\phi), I(t,\phi)\right)=\liminf_{t\to\infty} p(\Phi(t)\phi)\geq\eta,\quad \forall\phi\in M_0.$$

This completes the proof. $\square$

In the case where $d_2=0$, we see that $N(t)$ satisfies equation (4.6), and hence, $\lim_{t\to\infty} N(t)=N_e$. Thus, $I(t)$ satisfies the nonautonomous equation

$$I'(t)=\frac{\beta(t)(N(t)-I(t))I(t)}{N(t)}-(d+\gamma)I(t)$$

which is asymptotic to the following periodic equation

$$I'(t)=\frac{\beta(t)(N_e-I(t))I(t)}{N_e}-(d+\gamma)I(t). \tag{4.8}$$

If $R_0>1$, i.e., $\bar{\beta}>d+\gamma$, and $\beta(t)>0$, $\forall t\in[0,\omega]$, then [17, Theorem 5.2.1] implies that equation (4.8) has a unique positive ω-periodic solution $\bar{I}(t)$, which is globally asymptotically stable in $\mathbb{R}\setminus\{0\}$. By the theory of asymptotically periodic systems (see [17, Section 3.2]), it follows that $\lim_{t\to\infty}\big(I(t)-\bar{I}(t)\big)=0$, and hence $\lim_{t\to\infty}\big(S(t)-(N_e-\bar{I}(t))\big)=0$. This implies that system (4.5) has a globally attractive positive ω-periodic solution $\big(N_e-\bar{I}(t), \bar{I}(t)\big)$.

By applying the perturbation theory of a globally stable fixed point (see [10]) and the theorem on uniform persistence uniform in parameters (see [17, Theorem 1.4.2]) to the Poincaré map of system (4.5), we can further show that if $R_0 > 1$, $\beta(t) > 0$, $\forall t \in [0, \omega]$, and $d_2 \geq 0$ is sufficiently small, system (4.5) has a globally attractive positive ω-periodic solution $(\hat{S}(t), \hat{I}(t))$. However, it is a challenging problem to study the uniqueness and global attractivity of positive periodic solution of (4.5) in the case where $R_0 > 1$ and the parameter $d_2 \geq 0$ is not small.

Finally, we remark that it should be interesting to find applications of Theorems 3.2 and 3.3 to some practical models.

References

[1] K. Cooke, P. van den Driessche, and X. Zou, Interaction of maturation delay and nonlinear birth in population and epidemic models, J. Math. Biol., 39 (1999) 332-352.

[2] J. K. Hale, Asymptotic Behavior of Dissipative Systems, Mathematical Surveys and Monographs 25, Amer. Math. Soc., Providence, RI, 1988.

[3] J. K. Hale and S. M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[4] V. Hutson and K. Schmitt, Permanence and the dynamics of biological systems, Math. Biosci., 111 (1992) 293-326.

[5] Y. Kuang, Delay Differential Equations With Applications in Population Dynamics, Academic Press, 1993.

[6] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, SIAM J. Math. Anal., 37 (2005) 251-275.

[7] P. Massatt, Stability and fixed points of point dissipative systems, J. Differential Equations, 40 (1981) 217-231.

[8] G. R. Sell and Y. You, Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.

[9] H. L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, Mathematical Surveys and Monographs 41, Amer. Math. Soc., Providence, RI, 1995.

[10] H. L. Smith and P. Waltman, Perturbation of a globally stable steady state, Proc. Amer. Math. Soc., 127 (1999) 447-453.

[11] X. Tang, D. Cao and X. Zou, Global attractivity of positive periodic solution to periodic Lotka-Volterra systems with pure delay, J. Differential Equations, 228 (2006) 580-610.

[12] H. R. Thieme, Uniform persistence and permanence for non-autonomous semiflows in population biology, Math. Biosci., 166 (2000) 173-201,

[13] H. R. Thieme, Convergence results and Poincaré-Bendixson thichotomy for asymptotically autonomous differential equations, J. Math. Biol., 30 (1992) 755-763.

[14] L. Wang, Z. Wang and X. Zou, Periodic solutions of neutral functional differential equations, Journal London Math. Soc., 65 (2002) 439-452.

[15] D. Xu and X. -Q. Zhao, Dynamics in a periodic competitive model with stage structure, J. Math. Anal. Appl., 311 (2005) 417-438.

[16] X.-Q. Zhao, Uniform persistence and periodic coexistence states in infinite-dimensional periodic semiflows with applications, Canadian Appl. Math. Quart., 3 (1995) 473-495.

[17] X.-Q. Zhao, Dynamical Systems in Population Biology, Springer-Verlag, New York, 2003.

[18] X. -Q. Zhao and X. Zou, Threshold dynamics in a delayed SIS epidemic Model, J. Math. Anal. Appl., 257 (2001) 282-291.