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CONVERGENCE IN ASYMPTOTICALLY PERIODIC TRIDIAGONAL COMPETITIVE-COOPERATIVE SYSTEMS

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ABSTRACT. It is shown that there is no cyclic chain of fixed points of the Poincaré map associated with periodic tridiagonal competitive-cooperative systems. The convergence in asymptotically periodic tridiagonal competitive-cooperative systems is then obtained by appealing to asymptotically periodic semi-flow theory.

1. Asymptotically periodic semi-flows. In this preliminary section we summarize some general results on asymptotically periodic semi-flows. For their motivations and proofs, we refer to the recent paper [**20**].

Let (X, d) be a metric space. A continuous mapping $\Phi : \Delta \times X \to X$, $\Delta = \{(t, s); 0 \le s \le t < \infty\}$, is called a nonautonomous semi-flow if Φ satisfies the following properties:

(i) $\Phi(s, s, x) = x$ for all $s \ge 0$, $x \in X$;

(ii) $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x)$ for all $t \ge s \ge r \ge 0$.

Recall that $T(t) : X \to X$, $t \ge 0$, is called an ω-periodic semi-flow on X if there is an $\omega > 0$ such that $T(t)x$ is continuous in $(t, x) \in [0, \infty) \times X$, $T(0) = I$ and $T(t + \omega) = T(t)T(\omega)$ for all $t \ge 0$, see [**5, 18**]. For convenience, we also use the notation $T(t, x) = T(t)x$, $x \in X$, $t \ge 0$.

Definition 1.1. A nonautonomous semi-flow $\Phi : \Delta \times X \to X$ is called *asymptotically periodic with limit periodic semi-flow* $T(t) : X \to X$, $t \ge 0$, if

$$\Phi(t_j + n_j\omega, n_j\omega, x_j) \longrightarrow T(t)x, \quad j \to \infty,$$

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for any three sequences $t_j \to t$, $n_j \to \infty$, $x_j \to x$, $j \to \infty$, with $x, x_j \in X$.

Theorem 1.1 (Reduction). *Let $\Phi : \Delta \times X \to X$ be an asymptotically periodic semi-flow with limit ω-periodic semi-flow $T(t) : X \to X$, $t \geq 0$, and $T_n(x) = \Phi(n\omega, 0, x)$, $n \geq 0$, $x \in X$ and $S(x) = T(\omega)x$, $x \in X$. Assume that A_0 is a compact S-invariant subset of X. If, for some $y \in X$, $\lim_{n\to\infty} d(T_n(y), A_0) = 0$, then*

$$\lim_{t\to\infty} d(\Phi(t,0,y), T(t)A_0) = 0.$$

By a sequence of continuous mapping $S_m : X \to X$, $m \in \mathbf{N}$, we define a discrete dynamical process by

$$\begin{cases} T_n = S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0 : X \to X, & n \geq 1 \\ T_0 = I \end{cases}$$

where $I : X \to X$ is the identical mapping on X. For $x \in X$, we let $\gamma^+(x) = \{T_n(x); n \geq 0\}$ denote its orbit and $\omega(x) = \{y; y \in X$ and there is $n_k \to \infty$ such that $T_{n_k}(x) \to y$ as $k \to \infty\}$ its ω-limit set.

Definition 1.2. Let $T_n : X \to X$, $n \geq 0$, be a discrete dynamical process and $S : X \to X$ a continuous map. T_n, $n \geq 0$, is called *asymptotically autonomous with limit discrete semi-flow S^n, $n \geq 0$*, if

$$S_{m_j}(x_j) \longrightarrow S(x), \quad j \to \infty,$$

for any two sequences $m_j \to \infty$, $x_j \to x$, $j \to \infty$, with $x, x_j \in X$.

Let $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$. For any given strictly increasing continuous function $\phi : [0,\infty) \to [0,1)$ with $\phi(0) = 0$ and $\phi(\infty) = 1$, e.g., $\phi(s) = s/(1+s)$, we can define a metric ρ on $\overline{\mathbf{N}}$ as $\rho(m_1, m_2) = |\phi(m_1) - \phi(m_2)|$, for any $m_1, m_2 \in \overline{\mathbf{N}}$, and then $\overline{\mathbf{N}}$ is compactified. Let $\tilde{X} = \overline{\mathbf{N}} \times X$. We define a mapping $\tilde{S} : \tilde{X} \to \tilde{X}$ by

$$\tilde{S}((m,x)) = \begin{cases} (1+m, S_m(x)) & m < \infty, x \in X, \\ (\infty, S(x)) & m = \infty, x \in X. \end{cases}$$

By Definition 1.2, it easily follows that $\tilde{S} : \tilde{X} \to \tilde{X}$ is continuous. Then we can embed both $T_n : X \to X$, $n \geq 0$, and $S^n : X \to X$, $n \geq 0$, into a discrete semi-flow $\tilde{S}^n$, $n \geq 0$, on the larger metric space $\tilde{X}$ with

$$\tilde{S}^n((m,x)) = \begin{cases} (m+n, S_{m+(n-1)} \circ \cdots \circ S_{m+1} \circ S_m(x)) & m < \infty, \\ (\infty, S^n(x)) & m = \infty. \end{cases}$$

In particular, let $m = 0$,

$$\tilde{S}^n((0,x)) = (n, S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0(x)) = (n, T_n(x)), \quad n \geq 0.$$

Then, by the compactness of $\overline{\mathbf{N}}$, for any precompact orbit $\gamma^+(x)$ of T_n, $n \geq 0$, the orbit $\gamma^+((0,x))$ of $\tilde{S}^n$, $n \geq 0$, is precompact and

$$\{\infty\} \times \omega(x) = \omega((0,x))$$

where $\omega((0,x))$ is the ω-limit set of $(0,x)$ for $\tilde{S}_n$, $n \geq 0$, in the usual way.

Theorem 1.2 (Omega limit set). *Let the orbit $\gamma^+(x)$ of $T_n : X \to X$, $n \geq 0$, be precompact in X. Then its ω-limit set $\omega(x)$ has the following properties*:

(a) *$\omega(x)$ is nonempty and compact*;

(b) *$\omega(x)$ is S-invariant, i.e., $S(\omega(x)) = \omega(x)$, and compactly S-invariantly connected*;

(c) *$\omega(x)$ attracts $\gamma^+(x)$, i.e., $\lim_{n\to\infty} d(T_n(x), \omega(x)) = 0$*;

(d) *$\omega(x)$ is chain recurrent for S.*

Assertions (a), (b) and (c) of Theorem 1.2 were established in [**20**, Theorem 2.1]. Assertion (d) can also be derived by applying the above-mentioned embedding approach and the corresponding result for mappings, see, e.g., [**11**].

Theorem 1.3 (Butler-McGehee type lemma). *Let M be an isolated S-invariant set in X, and let $\gamma^+(x)$ be an orbit of T_n, $n \geq 0$, and $\omega(x)$ its ω-limit set. Assume that $\gamma^+(x)$ is precompact in X and that $\omega(x) \cap M \neq \varnothing$ but $\omega(x) \not\subseteq M$. Then*

(a) *there exists a* $u \in \omega(x)\backslash M$ *with its* S*-orbit* $\gamma_S^+(u) \subseteq \omega(x)$ *and* ω-S*-limit set* $\omega_S(u) \subseteq M$*, and*

(b) *there exists a* $w \in \omega(x)\backslash M$ *with a full* S*-orbit* $\gamma_S(\omega) \subseteq \omega(x)$ *and its* α-S*-limit set* $\alpha_S(w) \subseteq M$.

Theorem 1.4 (Attractivity). *Let* M *be a compact* S*-invariant subset of* X *which is locally asymptotically stable for* S *and* $W^s(M) = \{y \in X; \omega_S(y) \neq \varnothing$ *and* $\omega_S(y) \subseteq M\}$ *be its stable set. Then, for any precompact* T_n, $n \geq 0$*-orbit* $\gamma^+(x)$ *with* $\omega(x) \cap W^s(M) \neq \varnothing$, $\omega(x) \subseteq M$.

Theorem 1.5 (Convergence). *Assume that each fixed point of* S *is isolated, that there is no* S*-cyclic chain of fixed points of* S*, and that every precompact* S*-orbit converges to some fixed point of* S*. Then any precompact orbit* $\gamma^+(x)$ *of* T_n, $n \geq 0$*, converges to some fixed point of* S.

Theorem 1.6 (Uniform persistence/repellor). *Let* X_0 *and* ∂X_0 *be open and closed subsets of* X*, respectively, such that* $X_0 \cap \partial X_0 = \varnothing$ *and* $X = X_0 \cup \partial X_0$*. Assume that* $S_m(X_0) \subseteq X_0$ *for all* $m \geq 0$ *and* $S(X_0) \subseteq X_0$*, and that*

(1) *there is a compact* S*-invariant subset* A_0 *of* X_0 *which is globally asymptotically stable for* S *in* X_0;

(2) *Let* A_∂ *be the maximal compact invariant set of* S *in* ∂X_0. $\tilde{A}_\partial = \cup_{x \in A_\partial} \omega_S(x)$ *has an isolated and* S*-acyclic covering* $\cup_{i=1}^k M_i$ *in* ∂X_0*, that is,* $\tilde{A}_\partial \subseteq \cup_{i=1}^k M_i$*, where* $M_1, M_2, \ldots, M_k$ *are pairwise disjoint, compact and isolated invariant sets of* S *in* ∂X_0 *such that each* M_i *is also an isolated* S*-invariant set in* X*, and no subset of* M_i*'s forms a cycle for* $S_\partial = S|_{A_\partial}$ *in* A_∂;

(3) $\tilde{W}^s(M_i) \cap X_0 = \varnothing$, $i = 1, 2, \ldots, k$*, where* $\tilde{W}^s(M_i) = \{x; x \in X$*, the* $\omega - T_n$, $n \geq 0$*-limit set* $\omega(x) \neq \varnothing$ *and* $\omega(x) \subseteq M_i\}$.

Then for any precompact orbit $\gamma^+(x)$ *of* T_n, $n \geq 0$*, with* $x \in X_0$*, its* ω*-limit set* $\omega(x) \subseteq A_0$.

For an asymptotically periodic semi-flow $\Phi : \Delta \times X \to X$ with limit ω-periodic semi-flow $T(t) : X \to X$, $t \geq 0$, let $T_n(x) = \Phi(n\omega, 0, x)$, $n \in N$, $x \in X$ and $S = T(\omega) : X \to X$. Define $S_n : X \to X$,

$n \geq 0$, by $S_n(x) = \Phi((n+1)\omega, n\omega, x)$, $n \geq 0$, $x \in X$. Then, by the properties of nonautonomous semi-flows, $T_n(x) = S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0(x)$, $n \geq 1$, $x \in X$. By Definition 1.1, it then easily follows that $\lim_{(n,x)\to(\infty,x_0)} S_n(x) = S(x_0)$, i.e., $T_n : X \to X$, $n \geq 0$, is an asymptotically autonomous discrete dynamical process with limit autonomous discrete semi-flow $S^n : X \to X$, $n \geq 0$, in the sense of Definition 1.2.

Theorems 1.1, 1.3, 1.4 and 1.6 were applied in [**20**] for the global attractivity of scalar asymptotically periodic Kolmogorov parabolic equations and the uniform persistence of asymptotically periodic parabolic predator-prey systems, and in [**16**] for the qualitative analysis of periodically operated chemostat models via certain conservation principle. For the global attractivity in scalar asymptotically almost periodic Kolmogorov equations and its applications, we refer to the recent paper [**17**]. In this paper our main aim is to apply Theorems 1.1 and Theorem 1.5 to discuss the convergence in asymptotically periodic tridiagonal competitive-cooperative systems and its application to a cascade model of neural nets studied in [**8, 14**].

2. Periodic tridiagonal competitive-cooperative systems. Consider the nonlinear periodic tridiagonal system

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(t, y_1, y_2), \\ \dfrac{dy_j}{dt} = f_j(t, y_{j-1}, y_j, y_{j+1}), \quad 2 \leq j \leq n-1, \\ \dfrac{dy_n}{dt} = f_n(t, y_{n-1}, y_n), \end{cases} \tag{2.1}$$

where $f = (f_1, \dots, f_n)^T \in C'(R^{n+1}, R^n)$ is ω-periodic in t for some $\omega > 0$, i.e., $f(t+\omega, y) = f(t, y)$. We assume that there exists $\delta_i \in \{-1, +1\}$, $1 \leq i \leq n-1$, such that

$$\delta_i \frac{\partial f_i}{\partial y_{i+1}} > 0, \qquad \delta_i \frac{\partial f_{i+1}}{\partial y_i} > 0, \quad 1 \leq i \leq n-1, \tag{2.2}$$

holds for all $(t, y) \in R^{n+1}$.

The requirement (2.2) implies that the Jacobian matrix $\partial f/\partial y$ is tridiagonal and sign symmetric in the sense that $\partial f_i/\partial y_{i+1}$ and

$\partial f_{i+1}/\partial y_i$ have the same sign δ_i. In case $\delta_i = -1$ for all i, (2.1) is competitive, and in case $\delta_i = +1$ for all i, (2.1) is cooperative. As shown in [**13**], by a change of variables, we will always assume (2.1) is cooperative.

For simplicity, we further assume that all solutions of (2.1) exist globally for all $t \geq 0$. Let S be the Poincaré map associated with (2.1), i.e., $S(v) = y(\omega, v)$, $v \in R^n$, where $y(t,v)$ is the unique solution of (2.1) satisfying $y(0,v) = v$. We denote the set of all fixed points of S by Fix(S). In [**13**], it was shown that every bounded solution of (2.1) is asymptotic to an ω-periodic solution, i.e., the omega limit set of every bounded orbit for $\{S^n\}$ is a fixed point of S. In what follows, we will show that there is no cyclic chain of fixed points of S by using an integer-valued Lyapunov function. To do this, we need a series of lemmas.

Following [**12, 13**], we define a unique continuous function $\sigma : \Lambda \to \{0, 1, 2, \dots, n-1\}$ on

$$\Lambda = \{v \in R^n : v_1 \neq 0, v_n \neq 0 \text{ and if } v_i = 0 \text{ for some } i, 2 \leq i \leq n-1, \text{ then } v_{i-1}v_{i+1} < 0\}$$

such that for any $v \in \Lambda_0 = \{v \in R^n : V_i \neq 0 \text{ for all } 1 \leq i \leq n\}$,

$$\sigma(v) = \#\{i : v_i v_{i+1} < 0\},$$

where $\#$ denotes the cardinality of the set. Clearly, Λ is open and dense in R^n.

Consider the periodic linear system

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 \\ \dfrac{dx_j}{dt} = a_{jj-1}(t)x_{j-1} + a_{jj}(t)x_j + a_{jj+1}(t)x_{j+1}, \quad 2 \leq j \leq n-1, \\ \dfrac{dx_n}{dt} = a_{nn-1}(t)x_{n-1} + a_{nn}(t)x_n, \end{cases} \tag{2.3}$$

where the $a_{ij}(\cdot)$ are continuous and ω-periodic functions defined on R and

$$a_{jj+1}(t) > 0, \quad t \in R, \quad 1 \leq j \leq n-1,$$

and

$$a_{jj-1}(t) > 0, \quad t \in R, \quad 2 \le j \le n.$$

Lemma 2.1 [**13**, Theorem 1.3]. *Let $X(t)$ denote the fundamental matrix solution of the linear system (2.3) satisfying $X(0) = I$, where I is the identity matrix. Then (2.3) has n distinct positive Floquet multipliers $\alpha_1, \dots, \alpha_n$, satisfying*

$$\alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} > \alpha_n > 0.$$

If E_{α_i} are the corresponding one-dimensional eigenspaces of $X(\omega)$, then $E_{\alpha_i} \backslash \{0\} \subset \Lambda$ and

$$\sigma(E_{\alpha_i} \backslash \{0\}) = i - 1, \quad 1 \le i \le n.$$

Lemma 2.2 [**13**, Lemma 2.1]. *Let $y(t)$ and $\bar{y}(t)$ be distinct solutions of (2.1) on an interval I. Then $y(t) - \bar{y}(t) \in \Lambda$ except possibly at finitely many values of $t \in I$. $\sigma(y(t) - \bar{y}(t))$ is locally constant and strictly decreases as t increases through a value s at which it is not defined.*

Lemma 2.3. *Let $\psi \in \mathrm{Fix}\,(S)$, and let $k - 1 \ge 0$ be the number of eigenvalues of $S'(\psi)$ greater than one and $l \ge 0$ be the number of those not less than one.*

(a) *If $w_1, w_2 \in R^n$ with $w_1 \ne w_2$ and $\omega(w_1) = \omega(w_2) = \{\psi\}$, then $\sigma(w_1 - w_2) \ge k - 1$.*

(b) *If $w_1, w_2 \in R^n$ with $w_1 \ne w_2$ and $\alpha(w_1) = \alpha(w_2) = \{\psi\}$, then $\sigma(w_1 - w_2) \le l - 1$.*

Proof. Clearly, $l = k$ or $l = k - 1$. Let $v(t) = y(t, w_1) - y(t, w_2)$, $t \in R$. Then $u(t)$ satisfies the following nonautonomous equation

$$\frac{dv}{dt} = C(t)v, \quad v \in R^n, \tag{2.4}$$

where $C(t) = (c_{ij}(t))$ and

$$c_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial y_j}(t, u_{i-1}(s,t), u_i(s,t), u_{i+1}(s,t))\, ds$$

with $u_j(s,t) = sy_j(t, w_1) + (1-s)y_j(t, w_2)$, $j = i-1, i, i+1$.

Let $C_0(t) = (c^0_{ij}(t))$ with $c^0_{ij}(t) = (\partial f_i/\partial y_j)(t, y_{i-1}(t,\psi), y_i(t,\psi), y_{i+1}(t,\psi))$, $1 \le i,\ j \le n$. Then $C_0(t)$ is an $n \times n$ ω-periodic matrix. Let $X_\psi(t)$ be the fundamental matrix solution of the linear ω-periodic system

$$\frac{dz}{dt} = C_0(t)z, \quad z \in R^n, \tag{2.5}$$

satisfying $X_\psi(0) = I$. Then $S'(\psi) = X_\psi(\omega)$. By Lemma 2.1, $X_\psi(\omega)$ has n distinct positive eigenvalues $\alpha_1, \dots, \alpha_n$, satisfying

$$\alpha_1 > \alpha_2 > \cdots > \alpha_{k-1} > 1 \ge \alpha_k > \alpha_{k+1} > \cdots > \alpha_n > 0.$$

Moreover, let E_{α_i} be the corresponding one-dimensional eigenvalue of $X_\psi(\omega)$. Then

$$E_{\alpha_i}\backslash\{0\} \subset \Lambda \quad \text{and} \quad \sigma(E_{\alpha_i}\backslash\{0\}) = i-1, \quad 1 \le i \le n.$$

In case (a), since

$$\lim_{t\to+\infty}(y(t,w_1) - y(t,\psi)) = 0 = \lim_{t\to+\infty}(y(t,w_2) - y(t,\psi)),$$

$\lim_{t\to+\infty}(C(t)-C_0(t)) = 0$, i.e., (2.4) is asymptotic to (2.5) as $t \to +\infty$. By [**1**, Corollary B.3 and Theorem B.5], it then easily follows that there exists some $1 \le i \le n$ such that

$$\lim_{m\to+\infty} |v(m\omega)|^{1/m} = \alpha_i, \tag{2.6}$$

and

$$\lim_{m\to+\infty} \frac{v(m\omega)}{|v(m\omega)|} = \phi_i \tag{2.7}$$

with $\phi_i \in E_{\alpha_i}$ and $|\phi_i| = 1$. We further claim that $\alpha_i \le 1$ and hence $i \ge k$. Suppose $\alpha_i > 1$. Choose $\alpha_0 \in (1, \alpha_i)$. By (2.6), there exists $N > 0$, such that, for all $m \ge N$,

$$|v(m\omega)|^{1/m} > \alpha_0, \qquad \text{i.e., } |v(m\omega)| > \alpha_0^m.$$

Then, since $\lim_{m\to+\infty}\alpha_0^m = +\infty$, $\{v(m\omega)\}_{m=1}^{+\infty}$ is unbounded, which is a contradiction to the boundedness of $v(t)$ on $[0,+\infty)$. Since $\phi_i \in E_{\alpha_i}\setminus\{0\}$, $\sigma(\phi_i) = i-1$. Then (2.7) implies that there exists $N_0 > 0$ such that, for all $m \geq N_0$,

$$\sigma(v(m\omega)) = \sigma\left(\frac{v(m\omega)}{|v(m\omega)|}\right) = i-1,$$

and hence, by Lemma 2.2,

$$\sigma(w_1 - w_2) = \sigma(v(0)) \geq \sigma(v(m\omega)) = i-1 \geq k-1.$$

In case (b), since

$$\lim_{t\to-\infty}(y(t,w_1) - y(t,\psi)) = 0 = \lim_{t\to-\infty}(y(t,w_2) - y(t,\psi)),$$

$\lim_{t\to-\infty}|C(t)-C_0(t)| = 0$, i.e., (2.4) is asymptotic to (2.5) as $t\to-\infty$. By the counterparts of [**1**, Corollary B.3 and Theorem B.5] for backward sequence, see [**1**, Theorem B.9], we can show that there exists some $1 \leq j \leq n$ such that

$$\lim_{m\to-\infty}|v(m\omega)|^{1/m} = \alpha_j \geq 1 \quad \text{and} \quad \lim_{m\to-\infty}\frac{v(m\omega)}{|v(m\omega)|} = \phi_j, \tag{2.8}$$

with $\phi_j \in E_{\alpha_j}$ and $|\phi_j| = 1$. Therefore, $j \leq l$ and $\sigma(\phi_j) = j-1$. It follows that there exists $N_1 > 0$ such that, for all $m \leq -N_1$,

$$\sigma(v(m\omega)) = \sigma\left(\frac{v(m\omega)}{|v(m\omega)|}\right) = j-1 \leq l-1,$$

and hence, by Lemma 2.2,

$$\sigma(w_1 - w_2) = \sigma(v(0)) \leq \sigma(v(m\omega)) = j-1 \leq l-1.$$

This completes the proof. □

Lemma 2.4. *For any $\psi_1, \psi_2 \in \mathrm{Fix}(S)$ with $\psi_1 \neq \psi_2$, then $\psi_1 - \psi_2 \in \Lambda$.*

Proof. Clearly, $y(t,\psi_1)$ and $y(t,\psi_2)$ are two distinct ω-periodic solutions. Let $v(t) = y(t,\psi_1) - y(t,\psi_2)$, $t \in R$. By Lemma 2.2, $v(t) \in \Lambda$ except possibly at finitely many values of $t \in R$, and $\sigma(v(t))$ strictly decreases as t increases through a value s at which it is not defined. Therefore, there exists $m_0 > 0$ such that

$$v(t) \in \Lambda, \quad \text{for all} \quad |t| \geq m_0\omega.$$

Since $v(t)$ is ω-periodic, $\sigma(v(-m_0\omega)) = \sigma(v(m_0\omega))$. It follows that $v(t) \in \Lambda$ for all $t \in [-m_0\omega_0, m_0\omega]$. In particular, $v(0) = \psi_1 - \psi_2 \in \Lambda$. This completes the proof. $\square$

Now we are in a position to prove the main result in this section.

Theorem 2.1. *There is no cyclic chain of fixed points of S.*

Proof. We first claim that there is no homoclinic orbit for S. Assume that there exists $w \notin \mathrm{Fix}\,(S)$ such that $\omega(w) = \{\psi\}$ and $\alpha(w) = \{\psi\}$, $\psi \in \mathrm{Fix}\,(S)$. Let $k - 1 \geq 0$ be the number of eigenvalues of $S'(\psi)$ greater than one. For every integer m, let $w_1 = S^m w$ and $w_2 = S^{m+1}w$. Clearly, $\omega(w_1) = \omega(w_2) = \{\psi\}$ and $\alpha(w_1) = \alpha(w_2) = \{\psi\}$. Applying Lemma 2.3 to the pair w_1 and w_2, we obtain that

$$\sigma(S^m w - S^{m+1}w) \geq k - 1,$$

and

$$\sigma(S^m w - S^{m+1}w) \leq l - 1 \leq k - 1.$$

Therefore, for any integer m,

$$\sigma(S^m w - S^{m+1}w) = \sigma(y(m\omega, w) - y((m+1)\omega, w)) = k - 1.$$

Since $w \notin \mathrm{Fix}\,(S)$, $y(t+\omega, w)|_{t=0} = y(\omega, w) \neq w = y(t,w)|_{t=0}$, and hence $y(t+\omega, w)$ and $y(t,w)$ are two distinct solutions of (2.1) on R. By Lemma 2.2,

$$y(t+\omega, w) - y(t,w) \in \Lambda,$$

and

$$\sigma(y(t+\omega, w) - y(t,w)) = k - 1, \quad \text{for all } t \in R.$$

Let $P : R^n \to R$ be defined by $P(x_1, \dots, x_n) = x_1$. By the definition of Λ, $P(y(t+\omega, w)) \neq P(y(t, w))$ for any $t \in R$. Therefore, the sequence $P(S^m w) = P(y(m\omega, w))$, $-\infty < m < \infty$, is strictly monotone. It then follows that $\lim_{m\to\infty} P(S^m w) \neq \lim_{m\to-\infty} P(S^m w)$. On the other hand, since $\lim_{m\to\infty} S^m w = \psi = \lim_{m\to-\infty} S^m w$, $\lim_{m\to\infty} P(S^m w) = P(\psi) = \lim_{m\to-\infty} P(S^m w)$. This leads to a contradiction.

Assume that there exists a cyclic chain of fixed points of S, i.e., there exist $w_1, w_2, \dots, w_k \notin \mathrm{Fix}\,(S)$ and $\psi_1, \psi_2, \dots, \psi_k \in \mathrm{Fix}\,(S)$ such that

$$\alpha(w_i) = \psi_i, \qquad \omega(w_i) = \psi_{i+1}, \quad 1 \leq i \leq k,$$

with $\psi_{k+1} = \psi_1$. In what follows, we make the convention that $\psi_{k+i} = \psi_i$, $w_{k+i} = w_i$. By our claim above, $k \geq 2$ and all ψ_i's are distinct. Clearly, $S^m(w_i) = y(m\omega, w_i) \to \psi_{i+1}$ and $S^{-m}(w_i) = y(-m\omega, w_i) \to \psi_i$ as $m \to \infty$. By Lemma 2.4, $\psi_i - \psi_{i+1} \in \Lambda$ for all $1 \leq i \leq k$. Let $v_i(t) = y(t, w_i) - y(t, w_{i+1})$, $t \in R$. Then, by Lemma 2.2, $v_i(t) \in \Lambda$ except possibly at finitely many values of t in R, and for any fixed $t \in R$ with $v_i(t) \in \Lambda$,

$$\sigma(\psi_i - \psi_{i+1}) \geq \sigma(v_i(t)) \geq \sigma(\psi_{i+1} - \psi_{i+2}), \quad 1 \leq i \leq k. \tag{2.9}$$

Therefore

$$\begin{aligned} \sigma(\psi_1 - \psi_2) \geq \sigma(\psi_2 - \psi_3) \geq \cdots \\ \geq \sigma(\psi_{k+1} - \psi_{k+2}) = \sigma(\psi_1 - \psi_2), \end{aligned} \tag{2.10}$$

and hence

$$\sigma(\psi_i - \psi_{i+1}) = \sigma(\psi_1 - \psi_2), \quad 1 \leq i \leq k. \tag{2.11}$$

By (2.9), (2.11) and Lemma 2.2,

$$\begin{gathered} v_i(t) \in \Lambda \quad \text{and} \quad \sigma(v_i(t)) = \sigma(\psi_1 - \psi_2), \\ \text{for all } t \in R \text{ and } 1 \leq i \leq k, \end{gathered} \tag{2.12}$$

and hence,

$$P(v_i(t)) \neq 0, \quad t \in R, \quad 1 \leq i \leq k. \tag{2.13}$$

Since $\psi_i - \psi_{i+1} \in \Lambda$, $1 \le i \le k$, $P(\psi_i) \ne P(\psi_{i+1})$, $1 \le i \le k$. Without loss of generality, we assume that $P(\psi_1) > P(\psi_2)$. Since

$$\begin{aligned}\lim_{m\to\infty} P(S^{-m}w_1 - S^{-m}w_2) &= \lim_{m\to\infty} P(S^{-m}w_1) - \lim_{m\to\infty} P(S^{-m}w_2)\\ &= P(\psi_1) - P(\psi_2) > 0,\end{aligned}$$

there exists $N > 0$ such that, for any $m \ge N$, $P(v_1(-m\omega)) = P(S^{-m}w_1 - S^{-m}w_2) > 0$. Combined with (2.13), we see $P(v_1(t)) > 0$ for any $t \in R$ and, in particular, $P(S^m w_1 - S^m w_2) > 0$ for all $m \ge 0$. Therefore, let $m \to \infty$, since $P(\psi_2) \ne P(\psi_3)$, we have $P(\psi_2) > P(\psi_3)$. In the same argument as above, it follows that $P(\psi_3) > P(\psi_4)$. Repeating this argument, we get

$$P(\psi_1) > P(\psi_2) > P(\psi_3) > \cdots > P(\psi_k) > P(\psi_{k+1}) = P(\psi_1),$$

which leads to a contradiction. This completes the proof. □

Remark 2.1. The arguments following [**13**, Theorem 2.2] imply that our Theorem 2.1 holds under the assumption that every ω-periodic solution $p(t)$ is nondegenerate, i.e., one is not a Floquet multiplier. Clearly, this implies that $p(t)$ is hyperbolic by Lemma 2.1. As above, we have shown that Theorem 2.1 holds without any additional condition by a different approach.

3. Asymptotically periodic tridiagonal competitive-cooperative systems. Consider the nonautonomous system

$$\frac{dy}{dt} = F(t, y), \quad y \in R^n \tag{3.1}$$

where $F \in C(R^+ \times R^n, R^n)$ satisfies the condition on the uniqueness of solutions. We assume that

(C1) $f(t, y) = (f_1, \dots, f_n)^T$ is ω-periodic tridiagonal competitive-cooperative;

(C2) $\lim_{t\to\infty} |F(t, y) - f(t, y)| = 0$ uniformly for y in any bounded subset of R^n.

Furthermore, we assume that the solutions of (2.1) and (3.1) exist globally for all $t \ge 0$. Then we have the following result.

Theorem 3.1. *Let* (C1) *and* (C2) *hold. Assume that each ω-periodic solution of* (2.1) *is isolated. Then every bounded solution of* (3.1) *is asymptotic to some ω-periodic solution of* (2.1).

Proof. Let $\phi(t,s,y)$ and $\phi_0(t,s,y)$, $t \geq s \geq 0$, be the unique solutions of (3.1) and (2.1) satisfying $\phi(s,s,y) = y$ and $\phi_0(s,s,y) = y$, respectively. Let $T(t)y = \phi_0(t,0,y)$, $t \geq 0$. Then $T(t) : R^n \to R^n$, $t \geq 0$, is a periodic semi-flow. By an argument similar to that in the proof of [**10**, Proposition 1.1(A)], it easily follows that $\phi : \Delta \times R^n \to R^n$ is an asymptotic periodic semi-flow with limit $T(t)$ in the sense of Definition 1.1. Let $T_n(y) = \phi(n\omega, 0, y)$, $n \geq 0$, $y \in R^n$, and let $S = T(\omega)$. Then, by our assumption, each fixed point of S is isolated. By [**13**, Theorem 2.2], every precompact S-orbit converges to some fixed point of S. Moreover, by Theorem 2.1, there is no cyclic chain of fixed points of S. By Theorem 1.5, it follows that any precompact orbit $\gamma^+(y)$ of T_n, $n \geq 0$, converges to some fixed point y^* of S, i.e., $\lim_{n\to\infty} d(T_n(y), y^*) = 0$, and hence, by Theorem 1.1, $\lim_{t\to\infty} d(\phi(t,0,y), T(t)y^*) = 0$. Clearly, $T(t)y^* = \phi_0(t,0,y^*)$ is an ω-periodic solution. This completes the proof. □

Now we consider a cascade model of neural sets [**8, 14**]

$$
\begin{cases} \dfrac{dx}{dt} = g(x), & x \in R^m, \\ \dfrac{dy}{dt} = f(x,y), & y \in R^n, \end{cases} \tag{3.2}
$$

where g and f are C^1 functions on $R^m \times R^n$. We assume that for any fixed x, $f(x,y)$, as a function of y, is tridiagonal competitive-cooperative, see (2.2). We further assume that every solution of (3.2) exists globally for $t \geq 0$. Then we have the following result.

Theorem 3.2. *Let* $(x(t), y(t))$, $t \geq 0$, *be a bounded solution of* (3.2). *Suppose that the solution $x(t)$ of $dx/dt = g(x)$ is asymptotic to an ω-periodic solution $x^*(t)$. Assume that every ω-periodic solution of the reduced system $dy/dt = f(x^*(t), y)$ is isolated. Then $(x(t), y(t))$ is asymptotic to an ω-periodic solution of* (3.2).

Proof. Let $(x(t), y(t))$, $t \geq 0$, be a given bounded solution of (3.2).

Then $y(t)$, $t \geq 0$, is a bounded solution of the nonautonomous system

$$\frac{dy}{dt} = f(x(t), y), \quad y \in R^n. \tag{3.3}$$

Since $\lim_{t\to\infty}(x(t) - x^*(t)) = 0$, $\lim_{t\to\infty} |f(x(t), y) - f(x^*(t), y)| = 0$ uniformly for y in any bounded subset of R^n. Then (3.3) is an asymptotically periodic tridiagonal competitive-cooperative system. By our assumption, every ω-periodic solution of the ω-periodic tridiagonal competitive-cooperative system

$$\frac{dy}{dt} = f(x^*(t), y), \quad y \in R^n \tag{3.4}$$

is isolated. By Theorem 3.1, it follows that $y(t)$ is asymptotic to an ω-periodic solution $y^*(t)$ of (3.4), i.e., $\lim_{t\to\infty}(y(t) - y^*(t)) = 0$. Clearly, $(x^*(t), y^*(t))$ is an ω-periodic solution of (3.2) and $(x(t), y(t))$ is asymptotic to $(x^*(t), y^*(t))$. This completes the proof. □

Remark 3.1. The same conclusion as in Theorem 3.2 was proved in [**14**] under the assumption that every ω-periodic solution of (3.4) is nondegenerate (and hence hyperbolic). Since the hyperbolicity implies the isolatedness, Theorem 3.2 is a generalization of [**14**, Theorem 1]. Clearly, we may have the corresponding result for the convergent dynamics in the case where $(x^*(t), y^*(t))$ is an equilibrium of (3.2).

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