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GLOBAL ATTRACTIVITY IN A CLASS OF NONMONOTONE REACTION-DIFFUSION EQUATIONS WITH TIME DELAY

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ABSTRACT. The global attractivity of the positive steady state is established for a class of nonmonotone time-delayed reaction-diffusion equations subject to the Neumann boundary condition via a fluctuation method. This result is also applied to three population models.

1 Introduction There have been extensive investigations on nonlocal and time-delayed population models in order to study the effects of spatial diffusion and time delay on the evolutionary behavior of biological systems, see, e.g., Gourley and Wu's survey paper [**1**] and the references therein. When such a system admits the comparison principle, the theory of monotone semiflows and the comparison arguments can be employed to study spreading speeds and traveling waves in the case of a unbounded domain, and the global stability in the case of a bounded domain. However, these problems become very challenging for nonmonotone and nonlocal reaction-diffusion systems with time delays.

Recently, Yi and Zou [**10**] proved the global attractivity of the unique positive constant equilibrium for a local and time-delayed reaction-diffusion model subject to the Neumann boundary condition in a nonmonotone case by combining a dynamical systems argument and some subtle inequalities. It is natural to ask whether the global attractivity holds for the nonlocal version of this model and other nonmonotone and nonlocal reaction-diffusion population models. The purpose of the current paper is to prove the global attractivity of the positive constant equilibrium for a large class of nonmotone reaction-diffusion equations with time delay. It turns out that this global attractivity result holds regardless of the diffusion coefficient, the time delay, and the form of kernel functions. In

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particular, when the kernel function $k(x,y) = \delta(x-y)$, where $\delta(x)$ is the Dirac function on $\mathbb{R}^m$, the nonlocal equation reduces to a local one.

In order to prove our main result, we will use a fluctuation method, which was developed by Thieme and Zhao [**7**] for a nonlocal, delayed and diffusive predator-prey model. We should point out that a similar method was also used in [**2, 5, 8**] to prove the upward convergence in the property of spreading speeds. It is easy to see that this method also works for nonlocal reaction-diffusion equations with distributed delays. In this paper, we consider nonmonotone reaction-diffusion equations with a single time delay for the sake of convenience.

The rest of this paper is organized as follows. In Section 2, we present some preliminary results on the global attractor, the principal eigenvalue, global extinction and uniform persistence for the model system. Section 3 is devoted to the proof of the global attractivity of the positive constant equilibrium under appropriate assumptions. In Section 3, we provide three examples to illustrate the applicability of the main result.

2 Preliminaries Let Ω be a bounded and open subset of $\mathbb{R}^m$ with a smooth boundary $\partial\Omega$, and $\partial/\partial\nu$ be the differentiation in the direction of the outward normal ν to $\partial\Omega$. Let $\delta(x)$ be the Dirac function on $\mathbb{R}^m$. For any $x \in \Omega$, we can define a Dirac (or point) measure δ_x in the following way: for any subset B of Ω, $\delta_x(B) = 1$ if $x \in B$, and $\delta_x(B) = 0$ if $x \notin B$. Throughout this paper, we always use $\int_\Omega \delta(x-y)\phi(y)dy$ to denote the measure integral $\int_\Omega \phi\, d\delta_x$. It then follows that $\int_\Omega \delta(x-y)\phi(y)\,dy = \phi(x)$ for all $x \in \Omega$.

We consider the following time-delayed reaction-diffusion equation:

$$
\begin{cases}
\dfrac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - \alpha u(t,x)) \\
\qquad\qquad\qquad\qquad + \displaystyle\int_\Omega k(x,y) f(u(t-\tau,y))\,dy, \\
\dfrac{\partial u}{\partial \nu}(t,x) = 0, \qquad t > 0,\ x \in \partial\Omega, \\
u(t,x) = \phi(t,x) \geq 0, \quad t \in [-\tau, 0],\ x \in \Omega,
\end{cases}
\tag{2.1}
$$

where $d > 0$, $\tau \geq 0$, $\alpha > 0$, and Δ denotes the Laplacian operator on $\mathbb{R}^m$. Throughout this paper, we assume that the kernel function $k(x,y)$ has the following property:

(K) Either $k(x,y) = \delta(x-y)$, or $k(x,y)$ is a continuous and nonnegative function such that $\int_\Omega k(x,y)dy = 1$, $\forall x \in \Omega$, and the linear operator

$L(\phi)(x) := \int_\Omega k(x,y)\phi(y)dy$ is strictly positive on $C(\overline{\Omega},\mathbb{R})$ in the sense that $L(C(\overline{\Omega},\mathbb{R}_+)\setminus\{0\}) \subset C(\overline{\Omega},\mathbb{R}_+)\setminus\{0\}$.

Note that when $k(x,y) = \delta(x-y)$, (2.1) reduces to the following local time-delayed reaction-diffusion equation

$$(2.2)\qquad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - \alpha u(t,x)) + f(u(t-\tau,x)), \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \qquad\qquad t>0,\ x\in\partial\Omega, \\ u(t,x) = \phi(t,x) \geq 0, \quad t\in[-\tau,0],\ x\in\Omega. \end{cases}$$

We further impose the following assumptions on the function f.

(A1) $f:\mathbb{R}^+\to\mathbb{R}$ is Lipschitz continuous with $f(0)=0$ and $f'(0)>0$, and $f(u)\leq f'(0)u$ for all $u\geq 0$.
(A2) There exists a number $M>0$ such that for all $u>M$, $\bar{f}(u)<\alpha u$, where $\bar{f}(u) := \max_{v\in[0,u]} f(v)$.

Let $\mathbb{X} = C(\overline{\Omega},\mathbb{R})$ and $\mathbb{X}_+ = \{\phi\in\mathbb{X} : \phi(x)\geq 0, \forall x\in\overline{\Omega}\}$. Then $(\mathbb{X},\mathbb{X}_+)$ is a strongly ordered Banach space. It is well known that the differential operator $A := d\Delta$ generates a C^0-semigroup $T(t)$ on $\mathbb{X}$. Moreover, the standard parabolic maximum principle (see, e.g., [**4,** Corollary 7.2.3]) implies that the semigroup $T(t):\mathbb{X}\to\mathbb{X}$ is strongly positive in the sense that $T(t)(\mathbb{X}_+\setminus\{0\})\subset Int(\mathbb{X}_+), \forall t>0$.

Let $\mathbb{Y} := C([-\tau,0],\mathbb{X})$ and $\mathbb{Y}_+ := C([-\tau,0],\mathbb{X}_+)$. For the sake of convenience, we identify an element $\phi\in\mathbb{Y}$ as a function from $[-\tau,0]\times\overline{\Omega}$ to $\mathbb{R}$ defined by $\phi(s,x)=\phi(s)(x)$. For any function $y:[-\tau,b)\to\mathbb{X}$, where $b>0$, we define $y_t\in\mathbb{Y}, t\in[0,b)$, by $y_t(s) = y(t+s), \forall s\in[-\tau,0]$. Define $F:\mathbb{Y}_+\to\mathbb{X}$ by

$$F(\phi)(x) = -\alpha\phi(0,x) + \int_\Omega k(x,y)f(\phi(-\tau,y))\,dy, \quad \forall x\in\overline{\Omega},\ \phi\in\mathbb{Y}_+.$$

Then we can write (2.1) as an integral equation

$$\begin{cases} u(t) = T(t)\phi(0) + \int_0^t T(t-s)F(u_s)ds, \quad t>0, \\ u_0 = \phi\in\mathbb{Y}_+, \end{cases}$$

whose solutions are called mild solutions of (2.1).

Since $T(t) : \mathbb{X} \to \mathbb{X}$ is strongly positive, we have

$$\lim_{h\to 0^+} \operatorname{dist}(\phi(0) + hF(\phi), \mathbb{X}_+) = 0, \quad \forall \phi \in \mathbb{Y}_+.$$

By [**3,** Proposition 3 and Remark 2.4], for each $\phi \in \mathbb{Y}_+$, (2.1) has a unique non-continuable mild solution $u(t,\phi)$ with $u_0 = \phi$, and $u(t,\phi) \in \mathbb{X}_+$ for all $t \in (0, \sigma_\phi)$. Moreover, $u(t,\phi)$ is a classical solution of (2.1) for $t > \tau$.

By the same arguments as in the proof of [**9,** Theorem 2.1], We have the following result.

Lemma 2.1. *Let* (A1)–(A2) *hold. Then for any* $\phi \in \mathbb{Y}_+$*, a unique solution* $u(t,\phi)$ *of* (2.1) *globally exists on* $[0,\infty)$*,* $\limsup_{t\to\infty} u(t,x,\phi) \le M$ *uniformly for* $x \in \overline{\Omega}$*, and the solution semiflow* $\Phi(t) = u_t(\cdot) : \mathbb{Y}_+ \to \mathbb{Y}_+, t \ge 0$*, admits a connected global attractor.*

Linearizing (2.1) at the zero solution, we obtain the following linear time-delayed equation

$$\tag{2.3} \begin{cases} \partial_t v(t,x) = d\Delta v(t,x) - \alpha v(t,x) \\ \qquad\qquad\qquad + f'(0) \displaystyle\int_\Omega k(x,y) v(t-\tau, y)\, dy, \\ \dfrac{\partial v(t,x)}{\partial \nu} = 0, \qquad t > 0,\ x \in \partial\Omega, \\ v(s,x) = \phi(s,x) \ge 0, \quad s \in [-\tau, 0],\ x \in \Omega. \end{cases}$$

Associated with (2.3), there is a nonlocal elliptic eigenvalue problem of delay type

$$\tag{2.4} \begin{cases} \lambda v(x) = d\Delta v(x) - \alpha v(x) + f'(0) e^{-\lambda\tau} \\ \qquad\qquad \times \displaystyle\int_\Omega k(x,y) v(y) dy, \quad x \in \Omega, \\ \dfrac{\partial v(x)}{\partial \nu} = 0, \qquad x \in \partial\Omega. \end{cases}$$

The following result shows that the stability of the zero solution for (2.1) can be determined by the sign of $f'(0) - \alpha$.

Lemma 2.2. *For any* $\tau \ge 0$*, the eigenvalue problem* (2.4) *has a principal eigenvalue* $\bar{\lambda}_0$*, and* $\bar{\lambda}_0$ *has the same sign as* $f'(0) - \alpha$*.*

Proof. By the same arguments as in [**4,** Theorem 7.6.1], it follows that the nonlocal elliptic eigenvalue problem

$$\begin{cases} \lambda v(x) = d\Delta v(x) - \alpha v(x) + f'(0) \\ \qquad\qquad \times \displaystyle\int_\Omega k(x,y)v(y)\,dy, & x \in \Omega, \\ \dfrac{\partial v(x)}{\partial \nu} = 0, & x \in \partial\Omega \end{cases} \tag{2.5}$$

has a principal eigenvalue, denoted by λ_0, with a positive eigenfunction. Letting $\mu(x) = \mu$, $m(x) = f'(0)$ and replacing $\Gamma(x, y, \tau)$ with $k(x, y)$ in [**7,** Theorem 2.2], we then see that the problem (2.4) has a principal eigenvalue $\bar{\lambda}_0$, and $\bar{\lambda}_0$ has the same sign as λ_0. On the other hand, it is easy to see that $\lambda = f'(0) - \alpha$ is an eigenvalue of (2.5) with the positive eigenfunction $v(x) \equiv 1$. Thus, the uniqueness of the principal eigenvalue implies that $\lambda_0 = f'(0) - \alpha$.

By the same arguments as in [**9,** Theorem 3.1], together with Lemma 2.2, we have the following threshold type result on the global attractivity of the zero solution and uniform persistence for system (2.1).

Lemma 2.3. *Let* (A1)–(A2) *hold, and let* $u(t, x, \phi)$ *be the solution of* (2.1) *with* $\phi \in \mathbb{Y}_+$. *Then the following two statements are valid.*

(i) *If* $f'(0) < \alpha$, *then for any* $\phi \in \mathbb{Y}_+$, *we have* $\lim_{t\to\infty} u(t, x, \phi) = 0$ *uniformly for* $x \in \overline{\Omega}$.
(ii) *If* $f'(0) > \alpha$, *then* (2.1) *admits at least one spatially homogeneous steady state* $u^* \in (0, M]$, *and there exists* $\eta > 0$ *such that for any* $\phi \in \mathbb{Y}_+$ *with* $\phi(0, \cdot) \not\equiv 0$, *we have* $\liminf_{t\to\infty} u(t, x, \phi) \geq \eta$ *uniformly for* $x \in \overline{\Omega}$.

Note that in the case (ii) above, the function $G(u) := f(u) - \alpha u$ satisfies $G(0) = 0$, $G'(0) > 0$ and $G(M) \leq 0$. It then follows that $G(u^*) = 0$ for some $u^* \in (0, M]$, and hence, u^* is a spatially homogeneous steady state of (2.1).

3 Global attractivity In this section, we establish the global attactivity of the positive and spatially homogeneous steady state u^* for system (2.1) by the fluctuation method developed in [**7,** Section 4].

Motivated by [**2,** Section 2], we introduce the following additional condition on f.

(A3) $f'(0) > \alpha$, $\frac{f(u)}{u}$ is strictly decreasing for $u \in (0, M]$, and $f(u)$ has the property (P) that for any $v, w \in (0, M]$ satisfying $v \le u^* \le w$, $\alpha v \ge f(w)$ and $\alpha w \le f(v)$, we have $v = w$.

Note that if $f(u)$ is nondecreasing for $u \in [0, M]$, then $f(u)$ has the property (P). Indeed, for any $0 < v \le u^* \le w \le M$ with $\alpha v \ge f(w)$ and $\alpha w \le f(v)$, we have

$$\alpha u^* \le \alpha w \le f(v) \le f(u^*) \le f(w) \le \alpha v \le \alpha u^*,$$

which implies that $w = v = u^*$. Combining this observation and [**2,** Lemma 2.1] with $f(u)$ replaced by $f(u)/\alpha$, we then have the following result.

Lemma 3.1. *Either of the following three conditions is sufficient for the property* (P) *in condition* (A3) *to hold.*

(P0) $f(u)$ *is nondecreasing for* $u \in [0, M]$.
(P1) $uf(u)$ *is strictly increasing for* $u \in (0, M]$.
(P2) $f(u)$ *is nonincreasing for* $u \in [u^*, M]$, *and* $f\left(\frac{1}{\alpha} f(u)\right)/u$ *is strictly decreasing for* $u \in (0, u^*]$.

Now we are in a position to prove our main result.

Theorem 3.1. *Assume that* (A1), (A2) *and* (A3) *hold. Then for any* $\phi \in \mathbb{Y}_+$ *with* $\phi(0, \cdot) \not\equiv 0$, *we have* $\lim_{t\to\infty} u(t, x, \phi) = u^*$ *uniformly for* $x \in \overline{\Omega}$.

Proof. For any given $\phi \in \mathbb{Y}_+$ with $\phi(0, \cdot) \not\equiv 0$, let $\omega(\phi)$ be the omega limit set of the positive orbit through ϕ for the solution semiflow $\Phi(t)$. By Lemma 2.1, we have $\omega(\phi) \subset [0, M]_{\mathbb{Y}}$, where $[0, M]_{\mathbb{Y}} := \{\phi \in \mathbb{Y} : 0 \le \phi(\theta, x) \le M, \forall (\theta, x) \in [-\tau, 0] \times \overline{\Omega}\}$. Note that $[0, M]_{\mathbb{Y}}$ is positively invariant for the solution semiflow $\Phi(t)$. It then suffices to prove the global attractivity of u^* for all $\phi \in [0, M]_{\mathbb{Y}}$ with $\phi(0, \cdot) \not\equiv 0$.

Let $\phi \in [0, M]_{\mathbb{Y}}$ be given such that $\phi(0, \cdot) \not\equiv 0$. It is easy to see that $u(t, x) := u(t, x, \phi)$ satisfies

$$\begin{aligned} u(t, x) = e^{-\alpha t} & \int_\Omega \Gamma(t, x, y) u(0, y)\, dy \\ & + \int_0^t e^{-\alpha s} \int_\Omega \Gamma(s, x, y) \int_\Omega k(y, z) f(u(t - s - \tau, z)\, dz\, dy\, ds, \end{aligned}$$

where $\Gamma(t,x,y)$ is the Green function associated with the linear parabolic equation $\partial u/\partial t = d\Delta u$, $t>0$, $x\in\Omega$, subject to the Neumann boundary condition. Following [**5**], we define a function $g:[0,M]^2\to\mathbb{R}$ by

$$g(v,w)=\begin{cases}\min\{f(u):\, v\le u\le w\}, & \text{if } v\le w,\\ \max\{f(u):\, w\le u\le v\}, & \text{if } w\le v.\end{cases}\tag{3.1}$$

Then $g(v,w)$ is nondecreasing in $v\in[0,M]$ and nonincreasing in $w\in[0,M]$. Moreover, $f(u)=g(u,u)$, $\forall u\in[0,M]$, and $g(v,w)$ is continuous in $(v,w)\in[0,M]^2$ (see [**6,** Section 2]). Thus, we have

$$\begin{aligned}u(t,x)&=e^{-\alpha t}\int_\Omega \Gamma(t,x,y)u(0,y)\,dy+\int_0^t e^{-\alpha s}\int_\Omega \Gamma(s,x,y)\\ &\quad\times\int_\Omega k(y,z)g(u(t-s-\tau,z),u(t-s-\tau,z))\,dz\,dy\,ds.\end{aligned}$$

Let

$$u^\infty(x):=\limsup_{t\to\infty}u(t,x),\quad u_\infty(x):=\liminf_{t\to\infty}u(t,x),\quad \forall x\in\overline{\Omega}.$$

Then Lemma 2.3 (ii) implies that

$$M\ge u^\infty(x)\ge u_\infty(x)\ge\eta>0,\quad \forall x\in\overline{\Omega}.$$

By Fatou's lemma, we further get

$$u^\infty(x)\le\int_0^\infty e^{-\alpha s}\int_\Omega\Gamma(s,x,y)\int_\Omega k(y,z)g(u^\infty(z),u_\infty(z))\,dz\,dy\,ds.$$

Let

$$u^\infty:=\sup_{x\in\overline{\Omega}}u^\infty(x),\qquad u_\infty:=\inf_{x\in\overline{\Omega}}u_\infty(x).$$

Clearly, $M\ge u^\infty\ge u_\infty\ge\eta>0$. Since

$$\int_\Omega k(x,y)\,dy=1,\qquad \int_\Omega\Gamma(s,x,y)\,dy=1,\quad \forall s>0,\ x\in\Omega,$$

we have

$$u^\infty\le g(u^\infty,u_\infty)\int_0^\infty e^{-\alpha s}\,ds=\frac{g(u^\infty,u_\infty)}{\alpha}.\tag{3.2}$$

Similarly, we can obtain

$$u_\infty \geq \frac{g(u_\infty, u^\infty)}{\alpha}. \tag{3.3}$$

By the definition of the function g, we can find $v, w \in [u_\infty, u^\infty] \subset (0, M]$ such that

$$g(u^\infty, u_\infty) = f(v) \qquad \text{and} \qquad g(u_\infty, u^\infty) = f(w).$$

It then follows from (3.2) and (3.3) that

$$f(v) \geq \alpha u^\infty \geq \alpha v \quad \text{and} \quad f(w) \leq \alpha u_\infty \leq \alpha w, \tag{3.4}$$

and hence,

$$\frac{f(w)}{\alpha w} \leq 1 = \frac{f(u^*)}{\alpha u^*} \leq \frac{f(v)}{\alpha v}.$$

This, together with the strict monotonicity of $f(u)/u$ for $u \in (0, M]$, implies that $v \leq u^* \leq w$. By (3.2) and (3.3), we also have

$$f(v) \geq \alpha u^\infty \geq \alpha w \quad \text{and} \quad f(w) \leq \alpha u_\infty \leq \alpha v. \tag{3.5}$$

Thus, the property (P) implies that $v = w = u^*$. It then follows from (3.4) that $u^\infty = u^*$ and $u_\infty = u^*$. Since

$$u^\infty \geq u^\infty(x) \geq u_\infty(x) \geq u_\infty, \quad \forall x \in \overline{\Omega},$$

we have $u^\infty(x) = u_\infty(x) = u^*$, $\forall x \in \overline{\Omega}$. This implies that

$$\lim_{t\to\infty} u(t, x) = u^*, \quad \forall x \in \overline{\Omega}. \tag{3.6}$$

It remains to prove that $\lim_{t\to\infty} u(t, x) = u^*$ uniformly for $x \in \overline{\Omega}$. For any $\psi \in \omega(\phi)$, there exists a sequence $t_n \to \infty$ such that $\Phi(t_n)\phi \to \psi$ in $\mathbb{Y}$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} u(t_n + \theta, x, \phi) = \psi(\theta, x)$$

uniformly for $(\theta, x) \in [-\tau, 0] \times \overline{\Omega}$. It follows from (3.6) that $\psi(\theta, x) = u^*$, $\forall (\theta, x) \in [-\tau, 0] \times \overline{\Omega}$. Thus, we have $\omega(\phi) = \{u^*\}$, which implies that $u(t, \cdot, \phi)$ converges to u^* in $\mathbb{X}$ as $t \to \infty$.

Remark 3.1. Note that any solution of the time-delayed ordinary differential equation $u'(t) = -\alpha u(t) + f(u(t-\tau))$ is a spatially homogeneous solution of the time-delayed reaction-diffusion equation (2.1). Thus, Theorem 3.1 implies the global attractivity of the positive equilibrium for the former equation under the assumptions (A1)–(A3) on f.

It seems that the above fluctuation method does not apply to the case of the Dirichlet boundary condition. In this case, the positive solutions of the delay equation $u'(t) = -\alpha u(t) + f(u(t-\tau))$ are not solutions of the nonlocal time-delayed reaction-diffusion equation (2.1) any more. For the existence of positive x-dependent steady state under a general boundary condition, we refer to [**9,** Theorem 3.1 (ii)]. However, it remains an open problem to study the uniqueness and global attractivity of such a positive steady sate for nonmonotone equation (2.1) subject to the Dirichlet boundary condition.

4 Examples In this section, we present three examples of the growth function $f(u)$, as discussed in [**2,** Section 4], to illustrate the applicability of our main result.

First, we consider the logistic type function $f(u) = ru\,(1-u/k)$, $r > 0$, $k > 0$. Clearly, $f'(0) = r$, $\max_{u\in[0,k]} f(u) = f(k/2) = (rk)/4$, $f(u)/u = r(1-u/k)$ is strictly decreasing on $(0,k]$. Assume that $r > \alpha$. Then $u^* := k(1-\frac{\alpha}{r})$ is the unique positive constant equilibrium of (2.1). We choose $M = u^*$ if $r \in (\alpha, 2\alpha]$, and $M = \frac{rk}{4\alpha}$ if $r > 2\alpha$. It then easily follows that the property (P0) holds in the case where $r \in (\alpha, 2\alpha]$, and the property (P2) holds in the case where $r \in (2\alpha, 3\alpha]$. Thus, (A1), (A2) and (A3) are satisfied if $r \in (\alpha, 3\alpha]$. By Theorem 3.1, we then have the following result.

Proposition 4.1. *Let $f(u) = ru\,(1-u/k)$ with $k > 0$ and $r \in (\alpha, 3\alpha]$. Then the positive steady state $u^* = k(1-\alpha/r)$ attracts all positive solutions of* (2.1).

Next, we consider the Ricker type function $f(u) = pue^{-au}$, $p > 0$, $a > 0$. Clearly, $f'(0) = p$ and $f(u)/u = pe^{-au}$ is strictly decreasing on $(0,+\infty)$. It is easy to see that $\max_{u\in[0,+\infty)} f(u) = f(1/a) = p/(ae)$. Assume that $p > \alpha$. Then $u^* := (\ln(p/\alpha))/a$ is the unique positive constant equilibrium of (2.1). We choose $M = u^*$ if $p/\alpha \in (1, e]$, and $M = p/(\alpha a e)$ if $p/\alpha > e$. It then easily follows that the property (P0) holds in the case where $p/\alpha \in (1,e]$, and the property (P2) holds in the case where $p/\alpha \in (e, e^2]$. Thus, (A1), (A2) and (A3) are satisfied if $p/\alpha \in (1, e^2]$. Then Theorem 3.1 implies the following result.

Proposition 4.2. *Let $f(u) = pue^{-au}$ with $p > 0$, $a > 0$ and $1 < p/\alpha \leq e^2$. Then the positive steady state $u^* = (\ln(p/\alpha))/a$ attracts all positive solutions of* (2.1).

Remark 4.1. In the case where $k(x,y) = \delta(x-y)$, Proposition 4.2 implies the global attractivity of the positive steady state for the local time-delayed equation (2.2), which was proved in [**10,** Theorem 3.1] via a quite different method.

Finally, we consider the generalized Beverton-Holt type function $f(u) = \frac{pu}{q+u^m}$, $m > 0$, $p > 0$, $q > 0$. Clearly, $f'(0) = p/q$ and $f(u)/u = p/(q+u^m)$ is strictly decreasing on $(0,+\infty)$. Assume that $p/q > \alpha$. It is easy to see that $u^* := (p/\alpha - q)^{\frac{1}{m}}$ is the unique positive constant equilibrium of (2.1), and that $f(u)$ is strictly increasing on $[0,+\infty)$ in the case where $m \in (0,1]$. In the case where $m > 1$, we have

$$\max_{u\in[0,+\infty)} f(u) = f(\bar{u}) = \frac{p(m-1)\bar{u}}{qm}, \qquad \bar{u} := \left(\frac{q}{m-1}\right)^{\frac{1}{m}}.$$

We can further show that $f(u)$ is strictly increasing on $[0,u^*]$ if $m \in (1, p/(p-\alpha q)]$, that $uf(u)$ is strictly increasing on $[0,+\infty)$ if $m \in (0,2]$, and that $uf(u)$ is strictly increasing on $\left[0,(2q/(m-2))^{\frac{1}{m}}\right]$ if $m > 2$. We choose $M = u^*$ if $m \in (0,p/(p-\alpha q)]$, and $M = f(\bar{u})/\alpha$ if $m > p/(p-\alpha q)$. It then follows that (P0) holds if $m \in (0, p/(p-\alpha q)]$, and (P1) holds if either $m \in (0,2]$, or $m > \max(2, p/(p-\alpha q))$ and $f(\bar{u})/\alpha \leq (2q/(m-2))^{\frac{1}{m}}$. Thus, (A1), (A2) and (A3) are satisfied if either $m \in (0, \max(2, p/(p-\alpha q))]$, or $m > \max(2, p/(p-\alpha q))$ and $f(\bar{u})/\alpha \leq (2q/(m-2))^{\frac{1}{m}}$. As a consequence of Theorem 3.1, we have the following result.

Proposition 4.3. *Let $f(u) = pu/(q+u^m)$ with $m > 0$, $p > 0$, $q > 0$, $p/q > \alpha$, and either $m \in (0, \max(2, p/(p-\alpha q))]$, or $m > \max\left(2, \frac{p}{p-\alpha q}\right)$ and $f(\bar{u})/\alpha \leq (2q/(m-2))^{\frac{1}{m}}$. Then the positive steady state $u^* = (p/\alpha - q)^{\frac{1}{m}}$ attracts all positive solutions of* (2.1).

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