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Spreading speed and traveling waves for the diffusive logistic equation with a sedentary compartment

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Abstract. By applying the theory of asymptotic speeds of spread and traveling waves to the diffusive logistic equation with a sedentary compartment, we establish the existence of minimal wave speed for monotone traveling waves and show that it coincides with the spreading speed for solutions with initial functions having compact supports.

Keywords. Spreading speed; Traveling waves; The diffusive logistic equation; A sedentary compartment.

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1 Introduction

It is well known that the diffusive logistic or Verhulst equation is a scalar reaction diffusion equation with a simple hump nonlinearity (quadratic nonlinearity in the classical case). This equation describes the immigration of a species into a territory or the advance of an advantageous gene into a population. The equation provides the classical example for traveling fronts in parabolic equations, and it forms the nucleus of more complex multi-species models in ecology, pattern formation and epidemiology (see, e.g., [7]). In order to consider the case where the population individuals switch between mobile and stationary states during their lifetime, Lewis and Schmitz [4] presented and analysed the following reaction-diffusion model

$$\begin{cases} \partial_t v = D\Delta v - \mu v - \gamma_2 v + \gamma_1 w, \\ \partial_t w = rw(1 - w/K) - \gamma_1 w + \gamma_2 v, \end{cases} \tag{1.1}$$

where $v(t,x)$ and $w(t,x)$ are spatial densities of migrating and sedentary subpopulations, respectively, D is diffusion coefficient of migrating subpopulation, γ_1 and γ_2 are transition rates between two states. In model (1.1), the migrants have a positive mortality μ while the sedentary subpopulation reproduces (with the intrinsic growth rate r) and is subject to a finite carrying capacity K. The authors of [4] determined the minimal speed for traveling waves under the assumption that the emigration rate is less than the intrinsic growth rate for the sedentary class ($\gamma_1 < r$). Recently, Hadeler and Lewis [3] studied, among others, the spread rate for the system (1.1) in the general case by using the theory developed in [11, 5, 12]. We note that the existence and nonexistence of monotone traveling wave, and hence the existence of minimal wave speed, for system (1.1) need to be investigated further (see [3, Section 5.3]).

The purpose of the present paper is to use the theory developed in [1, 2, 8, 9, 10] for nonlinear integral equations to study the asymptotic speed of spread and monotone traveling waves of system (1.1). For convenience and other possible applications, we then consider the following general diffusive logistic equation with a sedentary compartment

$$\begin{cases} \partial_t v(t,x) = D\Delta v(t,x) - rv(t,x) + f(w(t,x)), \\ \partial_t w(t,x) = g(w(t,x)) + \beta v(t,x), \end{cases} \tag{1.2}$$

with initial conditions

$$v(0,x) = \phi_1(x) \geq 0,\ w(0,x) = \phi_2(x) \geq 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where D, r and β are positive constants, and the conditions on functions f and g are to be specified in section 3.

This paper is organized as follows. In section 2, we present some preliminary results based on the paper [10]. In section 3, we first reduce system (1.2)–(1.3) into an integral equation, and then obtain the asymptotic speed of spread under appropriate assumptions. Section 4 is devoted to the existence and nonexistence of monotone traveling wave solutions. Our results show that the asymptotic speed of spread is exactly the minimal wave speed for monotone traveling waves.

2 Preliminaries

In this section, we present the preliminary results that will be used in the subsequent sections.

Consider nonlinear integral equations

$$u(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^n} F(u(t-s,x-y),s,y)\,dyds, \tag{2.1}$$

where $F : \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u and Borel measurable in (s,y), and $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is Borel measurable and bounded. Assume that

(A) There exists a function $k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that

(A1) $k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s,x)\,dxds < \infty$.

(A2) $0 \le F(u,s,x) \le uk(s,x), \forall u,s \ge 0, x \in \mathbb{R}^n$.

(A3) For every compact interval I in $(0,\infty)$, there exists some $\varepsilon > 0$ such that

$$F(u,s,x) \ge \varepsilon k(s,x), \quad \forall u \in I, s \ge 0, x \in \mathbb{R}^n.$$

(A4) For every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$F(u,s,x) \ge (1-\varepsilon)uk(s,x), \quad \forall u \in [0,\delta], s \ge 0, x \in \mathbb{R}^n.$$

(A5) For every $w > 0$, there exists some $\Lambda > 0$ such that

$$|F(u,s,x) - F(v,s,x)| \le \Lambda |u-v| k(s,x),\ \forall u,v \in [0,w], s \ge 0, x \in \mathbb{R}^n.$$

To obtain asymptotic properties of the solutions of equation (2.1), we make a couple of assumptions concerning k.

(B) $k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a Borel measurable function such that

(B1) $k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s,y)\,dyds \in (1,\infty)$.

(B2) There exists some $\lambda^\diamondsuit > 0$ such that

$$\int_0^\infty \int_{\mathbb{R}^n} e^{\lambda^\diamondsuit y_1} k(s,y)\,dyds < \infty,$$

where y_1 is the first coordinate of y.

(B3) There exist numbers $\sigma_2 > \sigma_1 > 0, \rho > 0$ such that

$$k(s,x) > 0, \quad \forall s \in (\sigma_1, \sigma_2), |x| \in [0, \rho).$$

(B4) k is isotropic.

Here a function $k : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is said to be isotropic if for almost all $s > 0$, $k(s,x) = k(s,y)$ whenever $|x| = |y|$. For a fixed $z \in \mathbb{R}^n$ with $|z| = 1$, define

$$\mathcal{K}(c,\lambda) := \int_0^\infty \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda(cs - z\cdot y)} k(s,y)\, dyds, \quad \forall c \geq 0,\ \lambda \geq 0,$$

where $\cdot$ means the usual inner product on $\mathbb{R}^n$. Assume that k is isotropic. Since for every $z \in \mathbb{R}^n$ with $|z| = 1$, there exists an orthogonal matrix A with $Az = -e_1$, where e_1 is the first canonical basis vector of $\mathbb{R}^n$, there holds

$$\mathcal{K}(c,\lambda) = \int_0^\infty \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda(cs + y_1)} k(s,y)\, dyds,$$

where y_1 is the first coordinate of y. If (B) holds, then for every $c > 0$, there exists some $\lambda^{\#}(c) \in (0,\infty]$ such that $\mathcal{K}(c,\lambda) < \infty$ for $\lambda \in [0, \lambda^{\#}(c))$ and $\mathcal{K}(c,\lambda) = \infty$ for $\lambda \in (\lambda^{\#}(c), \infty)$ ([8, Lemma 3.7]). Define

$$c^* := \inf\{c \geq 0 : \mathcal{K}(c,\lambda) < 1 \text{ for some } \lambda > 0\}.$$

The following result is useful for the computation of c^*.

Proposition 2.1. ([10, Proposition 2.3]) *Let (B) hold and assume that*

$\liminf_{\lambda \nearrow \lambda^{\#}(c)} \mathcal{K}(c,\lambda) \geq k^*$ *for every* $c > 0$. *Then there exists a unique* $\lambda^* \in (0, \lambda^{\#}(c^*))$ *such that* $\mathcal{K}(c^*, \lambda^*) = 1$ *and* $\mathcal{K}(c^*, \lambda) > 1$ *for* $\lambda \neq \lambda^*$. *Moreover,* c^* *and* λ^* *are uniquely determined as the solutions of the* *system*

$$\mathcal{K}(c,\lambda) = 1, \ \frac{d}{d\lambda}\mathcal{K}(c,\lambda) = 0.$$

Definition 2.1. A number $c^* > 0$ is called the asymptotic speed of spread for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ if $\lim_{t\to\infty,\,|x|\geq ct} u(t,x) = 0$ for every $c > c^*$, and there exists some $\bar{u} > 0$ such that $\lim_{t\to\infty,\,|x|\leq ct} u(t,x) = \bar{u}$ for every $c \in (0, c^*)$.

Definition 2.2. A function $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be admissible if for every $c, \lambda > 0$ with $\mathcal{K}(c,\lambda) < 1$, there exists some $\gamma > 0$ such that $u_0(t,x) \leq \gamma e^{\lambda(ct-|x|)}, \forall t \geq 0, x \in \mathbb{R}^n$.

The following two results show that c^* defined above is the asymptotic speed of spread for solutions of (2.1).

Theorem 2.1. ([10, Theorem 2.1]) *Let (A) and (B) hold and let $u(t,x)$ be a solution of (2.1) with $u_0(t,x)$ being admissible. Then $\lim_{t\to\infty,\,|x|\geq ct} u(t,x) = 0$ for each $c > c^*$.*

Theorem 2.2. ([10, Theorem 2.4]) *Let (A) and (B) hold and let $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be a bounded and Borel measurable function with the property that $u_0(t,x) \geq \eta > 0, \forall t \in (t_1, t_2), |x| \leq \eta$, for appropriate $t_2 > t_1 \geq 0$ and $\eta > 0$. Also, let u be a bounded solution of (2.1) and $u^\infty := \limsup_{t\to\infty} \sup_{x\in\mathbb{R}^n} u(t,x)$. Assume that $F(\,\cdot\,, s, x)$ is monotone increasing on $[0, u^\infty]$ for each $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $\lim_{t\to\infty} u_0(t,x) = 0$ uniformly in $x \in \mathbb{R}^n$. Let $u^* > 0$ be such that $\tilde{F}(u) := \int_0^\infty \int_{\mathbb{R}^n} F(u,s,y)\,dyds > u$ whenever $u \in (0, u^*)$ and $\tilde{F}(u) < u$ whenever $u \in (u^*, u^\infty]$. Then we have $\lim_{t\to\infty,|x|\leq ct} u(t,x) = u^*$, $\forall c \in (0, c^*)$.*

Next we consider the limiting equation of (2.1) with $n = 1$

$$u(t,x) = \int_0^\infty \int_{\mathbb{R}} F(u(t-s, x-y), s, y)\,dyds. \tag{2.2}$$

A solution $u(t,x)$ of (2.2) is said to be a traveling wave solution if it is of the form $u(t,x) = U(x+ct)$. The parameter c is called the wave speed, and the function $U(\cdot)$ is called the wave profile. Here, we require the following conditions on the wave profile:

$$U(\cdot) \text{ is positive and bounded on } \mathbb{R}, \text{ and } \lim_{\xi\to-\infty} U(\xi) = 0. \tag{2.3}$$

The following two results deal with the existence and nonexistence of traveling wave solutions of (2.2).

Theorem 2.3. ([10, Theorem 3.3]) *Let (A2) and (B) with* $n = 1$ *hold. Assume that there exists some* $u^* > 0$ *such that* $\tilde{F}(u^*) = u^*$ *and* $\tilde{F}(u) > u$ *for all* $u \in (0,u^*)$*, where* $\tilde{F}(u) := \int_0^\infty \int_{\mathbb{R}} F(u,s,y)\,dyds$*. Moreover, suppose that* $F(\,\cdot\,,s,x)$ *is increasing on* $[0,u^*]$ *for each* $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$*, and* $F(u,s,x) \geq (u - bu^\sigma)k(s,x), \forall u \in [0,\delta], (s,x) \in \mathbb{R}_+ \times \mathbb{R}$*, for appropriate* $\delta \in (0,u^*], \sigma > 1$ *and* $b > 0$*. Then for each* $c > c^*$*, there exists a monotone traveling wave solution of (2.2) with speed* c *and connecting* 0 *and* u^**.*

Theorem 2.4. ([10, Theorem 3.5]) *Let (A) and (B) hold. Then for each* $c \in (0,c^*)$*, there exists no traveling wave solution of (2.2) and (2.3) with speed* c*.*

Finally, we consider nonlinear integral equations

$$u(t,x) = u_0(t,x) + \int_0^t e^{-as} f_0(u(t-s,x))ds + \int_0^t \int_{\mathbb{R}^n} F_0\big(u(t-s,x-y),s,y\big)dyds \tag{2.4}$$

where $a > 0$, $f_0 \in C(\mathbb{R}_+,\mathbb{R})$, $F_0 : \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u and Borel measurable in (s,y), and $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is Borel measurable and bounded. We assume that

(H1) $f_0 \in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $f_0'(u) \geq 0$ and $f_0(u) \leq f_0'(0)u$ for all $u \geq 0$.

(H2) $F_0(u,s,x)$ satisfies (A1)-(A5), and the associated $k_0(s,x)$ satisfies (B2)-(B4).

Using the measure integral for Dirac function $\delta(x)$ on $\mathbb{R}^n$, we write equation (2.4) as

$$\begin{aligned} u(t,x) = u_0(t,x) \quad &+ \quad \int_0^t \int_{\mathbb{R}^n} e^{-as} f_0(u(t-s, x-y))\delta(y) dy ds \\ &+ \quad \int_0^t \int_{\mathbb{R}^n} F_0\big(u(t-s, x-y), s, y\big) dy ds. \end{aligned}$$

It then follows that (2.4) can be written formally as the equation (2.1) with

$$F(u,s,x) := f_0(u)e^{-as}\delta(x) + F_0(u,s,x), \quad k(s,x) := f_0'(0)e^{-as}\delta(x) + k_0(s,x).$$

Remark 2.1 *By modifying slightly the proofs of [8, Theorem 2.8 (c)], [10, Propositions 2.2 and 2.4], and [10, Theorems 2.1, 2.4, 3.3 and 3.5], we see that Theorems 2.1–2.4 in this section remain valid for equation (2.4) provided that assumptions (H1), (H2) and (B1) hold. Note that in all integral computations it is understood that* $\int_{\mathbb{R}^n} \phi(x-y)\delta(y)dy = \phi(x)$.

3 The spreading speed

Motivated by the biological model (1.1), we impose the following conditions on equation (1.2).

(C1) $f: \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous and nondecreasing, differentiable at 0, $f(0) = 0$, $f(u) > 0, \forall u > 0$, and f is sublinear on $\mathbb{R}_+$ in the sense that $f(\theta w) \geq \theta f(w)$ for any $\theta \in (0,1), w \in \mathbb{R}_+$.

(C2) $g : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, $g(0) = 0$, strictly sublinear on $\mathbb{R}_+$ in the sense that $g(\theta w) > \theta g(w)$ for any $\theta \in (0,1), w > 0$.

(C3) $\beta f'(0) + rg'(0) > 0$, and there exists $w^* > 0$ such that $rg(w^*) + \beta f(w^*) = 0$.

Consider the reaction system associated with (1.2)

$$\begin{cases} \frac{dv}{dt} = -rv + f(w), \\ \frac{dw}{dt} = g(w) + \beta v. \end{cases} \tag{3.1}$$

Because of assumptions (C1)–(C3) on f and g, system (3.1) is cooperative on $\mathbb{R}^2_+$, and admits a positive equilibrium $(\frac{f(w^*)}{r}, w^*)$. Also, two roots of the characteristic equation associated with the linearization at zero equilibrium of (3.1) are

$$\lambda_\pm = \frac{g'(0) - r \pm \sqrt{[g'(0) - r]^2 + 4[\beta f'(0) + rg'(0)]}}{2},$$

and hence, $\lambda_+ > 0$ and $\lambda_- < 0$. It is easy to see that every solution to (3.1) with nonnegative initial value remains nonnegative. By [14, Corollary 3.2], system (3.1) admits a unique steady state $(\frac{f(w^*)}{r}, w^*)$, which is globally asymptotically stable in $\mathbb{R}^2_+ \backslash \{0\}$. By the standard comparison arguments, it follows that solutions to (3.1) are uniformly bounded on $\mathbb{R}^2_+$.

Let $\mathbb{X} := BUC(\mathbb{R}^n, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^2$ with the usual supreme norm. Define

$$\mathbb{X}_+ = \{(\phi_1, \phi_2) \in \mathbb{X} : \phi_i(x) \geq 0, \ \forall x \in \mathbb{R}^n, i = 1, 2\}.$$

Then $\mathbb{X}_+$ is a positive cone of $\mathbb{X}$, and its induced partial ordering makes $\mathbb{X}$ into a Banach lattice. By using the theory developed in [6] (see, e.g., [13, Lemma 3.1]), we have the following result.

Lemma 3.1. *Let (C1)–(C3) hold. For any $\phi \in \mathbb{X}_+$, system (1.2) has a unique, bounded and nonnegative mild solution $U(t,x,\phi) = (v(t,x,\phi), w(t,x,\phi))$ with $U(0,\cdot,\phi) = \phi$, and the solution semiflow associated with (1.2) is monotone on $\mathbb{X}_+$.*

In the rest of this section, we will find the spreading speed c^* for solutions of system (1.2). In order to use the theory in [10], we need to reduce (1.2)–(1.3) into a scalar integral equation. Let $\Gamma(t, x-y)$ be the Green function associated with the parabolic equation

$$\begin{cases} \partial_t u = D\Delta u, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}^n, t > 0. \end{cases}$$

In the case where $n = 1$, we have

$$\Gamma(t, x-y) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-y)^2}{4Dt}\right).$$

Then $\partial_t v = D\Delta v - rv$ generates a linear semigroup $T(t) : BUC(\mathbb{R}^n, \mathbb{R}) \to BUC(\mathbb{R}^n, \mathbb{R})$, which is defined by

$$(T(t)\phi)(x) = \mathrm{e}^{-rt} \int_{\mathbb{R}^n} \Gamma(t, x-y)\phi(y)\, dy, \quad \forall \phi \in BUC(\mathbb{R}^n, \mathbb{R}). \tag{3.2}$$

Integrating the first equation of system (1.2), we have the following abstract integral form

$$v(t) = T(t)v(0) + \int_0^t T(t-s) f(w(s))\, ds,$$

that is,

$$v(t,x) = \mathrm{e}^{-rt} \int_{\mathbb{R}^n} \Gamma(t, x-y)\phi_1(y)\, dy \\ + \int_0^t \mathrm{e}^{-r(t-s)} \int_{\mathbb{R}^n} \Gamma(t-s, x-y) f(w(s,y))\, dyds. \quad (3.3)$$

Given $\alpha > 0$, we define a nondecreasing function $g_\alpha(\,\cdot\,)$ on $\mathbb{R}_+$ by

$$g_\alpha(w) = \sup\{\alpha u + g(u) : 0 \le u \le w\}, \quad \forall w \ge 0.$$

For every $M > 0$, we can choose $\alpha = \alpha(M) > 0$ so large that $\alpha w + g(w)$ is monotone increasing for $w \in [0, M]$, and hence, $g_\alpha(w) = \alpha w + g(w), \forall w \in [0, M]$. Thus, for any bounded solution of (1.2), we can choose sufficiently large $\alpha > 0$ such that the second equation in system (1.2) takes the form

$$\partial_t w(t,x) = -\alpha w(t,x) + g_\alpha(w(t,x)) + \beta v(t,x). \quad (3.4)$$

It follows from (3.4) that

$$\begin{aligned} w(t,x) &= \mathrm{e}^{-\alpha t}\phi_2(x) + \int_0^t \mathrm{e}^{-\alpha(t-s)}[g_\alpha(w(s,x)) + \beta v(s,x)]\, ds \\ &= \mathrm{e}^{-\alpha t}\phi_2(x) + \int_0^t \mathrm{e}^{-\alpha(t-s)} \int_{\mathbb{R}^n} \delta(x-y) g_\alpha(w(s,y))\, dyds \\ &\quad + \beta \int_0^t \mathrm{e}^{-\alpha(t-s)} v(s,x)\, ds, \end{aligned} \quad (3.5)$$

where $\delta(x)$ is the Dirac function. After a substitution, we have

$$\begin{aligned} &\int_0^t ds \mathrm{e}^{-\alpha(t-s)} \int_{\mathbb{R}^n} \delta(x-y) g_\alpha(w(s,y))\, dy \\ &\quad = \int_0^t ds_1 \mathrm{e}^{-\alpha s_1} \int_{\mathbb{R}^n} \delta(x-y) g_\alpha(w(t-s_1,y))\, dy \\ &\quad = \int_0^t ds \int_{\mathbb{R}^n} k_1(s, x-y) g_\alpha(w(t-s,y))\, dy, \end{aligned} \quad (3.6)$$

where $k_1(s,x) = \mathrm{e}^{-\alpha s}\delta(x), \forall x \in \mathbb{R}^n$ and $\forall s \geq 0$. By (3.3), we obtain

$$\int_0^t \mathrm{e}^{-\alpha(t-s)} v(s,x)\, ds = \int_0^t ds \mathrm{e}^{-\alpha(t-s)} \mathrm{e}^{-rs} \int_{\mathbb{R}^n} \Gamma(s, x-y)\phi_1(y)\, dy + G(t,x) \tag{3.7}$$

with

$$G(t,x) := \int_0^t ds \mathrm{e}^{-\alpha(t-s)} \int_0^s d\tau \mathrm{e}^{-r(s-\tau)} \int_{\mathbb{R}^n} \Gamma(s-\tau, x-y) f(w(\tau,y))\, dy.$$

Changing the order of the integrations in the expression $G(t,x)$, we have

$$\begin{aligned} G(t,x) &= \int_0^t d\tau \int_\tau^t ds \mathrm{e}^{-\alpha(t-s)} \mathrm{e}^{-r(s-\tau)} \int_{\mathbb{R}^n} \Gamma(s-\tau, x-y) f(w(\tau,y))\, dy \\ &= \int_0^t d\tau \int_{\mathbb{R}^n} dy \int_\tau^t ds \mathrm{e}^{-\alpha(t-s)} \mathrm{e}^{-r(s-\tau)} \Gamma(s-\tau, x-y) f(w(\tau,y)). \end{aligned}$$

After substitutions,

$$\begin{aligned} G(t,x) &= \int_0^t d\tau \int_{\mathbb{R}^n} dy f(w(\tau,y)) \int_0^{t-\tau} ds_1 \mathrm{e}^{-\alpha(t-\tau-s_1)} \mathrm{e}^{-rs_1} \Gamma(s_1, x-y) \\ &= \int_0^t d\tau \int_{\mathbb{R}^n} dy f(w(\tau,y)) \mathrm{e}^{-\alpha(t-\tau)} \int_0^{t-\tau} \mathrm{e}^{(\alpha-r)s_1} \Gamma(s_1, x-y)\, ds_1 \\ &= \int_0^t d\tau \int_{\mathbb{R}^n} k_2(t-\tau, x-y) f(w(\tau,y))\, dy \\ &= \int_0^t ds \int_{\mathbb{R}^n} k_2(s, x-y) f(w(t-s,y))\, dy, \end{aligned} \tag{3.8}$$

where $k_2(s,x) = \mathrm{e}^{-\alpha s} \int_0^s \mathrm{e}^{(\alpha-r)s_1} \Gamma(s_1, x)\, ds_1, \forall x \in \mathbb{R}^n$ and $\forall s \geq 0$.

Inserting (3.6)–(3.8) into (3.5), we obtain

$$\begin{aligned} w(t,x) &= \mathrm{e}^{-\alpha t}\phi_2(x) + \int_0^t ds \int_{\mathbb{R}^n} k_1(s, x-y) g_\alpha(w(t-s,y))\, dy \\ &\quad + \beta \bigg[\int_0^t ds \mathrm{e}^{-\alpha(t-s)} \mathrm{e}^{-rs} \int_{\mathbb{R}^n} \Gamma(s, x-y)\phi_1(y)\, dy \\ &\quad + \int_0^t ds \int_{\mathbb{R}^n} k_2(s, x-y) f(w(t-s,y))\, dy \bigg] \\ &= w_0(t,x) + \int_0^t \int_{\mathbb{R}^n} F_\alpha(w(t-s, x-y), s, y)\, dy ds, \end{aligned} \tag{3.9}$$

where

$$w_0(t,x) = \mathrm{e}^{-\alpha t}\phi_2(x) + \beta\int_0^t ds\mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{-rs}\int_{\mathbb{R}^n}\Gamma(s,x-y)\phi_1(y)\,dy \tag{3.10}$$

and

$$F_\alpha(w,s,y) = g_\alpha(w)k_1(s,y) + \beta f(w)k_2(s,y). \tag{3.11}$$

Let $\alpha + g'(0) > 0$. In view of (3.11), we define

$$k(s,y) := g'_\alpha(0)k_1(s,y) + \beta f'(0)k_2(s,y). \tag{3.12}$$

By conditions (C1) and (C2) and [10, Lemma 4.1], it follows that assumption (A) holds for (3.9).

Next, we need to compute some Laplace-like transforms of integral kernels. For any function $\phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{K}_\phi(c,\lambda) := \int_0^\infty\int_{\mathbb{R}^n}\mathrm{e}^{-\lambda(cs+y_1)}\phi(s,y)\,dyds, \quad c,\lambda \geq 0,$$

where y_1 is the first coordinate of y. By [10, Proposition 4.2], we have

$$\mathcal{K}_{k_1}(c,\lambda) = \int_0^\infty\int_{\mathbb{R}^n}\mathrm{e}^{-\lambda(cs+y_1)}\mathrm{e}^{-\alpha s}\delta(y)\,dyds = \int_0^\infty \mathrm{e}^{-(\lambda c+\alpha)s}\,ds = \frac{1}{\lambda c+\alpha},$$

and in the case where $\lambda^2 D - \lambda c - r < 0$,

$$\begin{aligned}
\mathcal{K}_{k_2}(c,\lambda) &= \int_0^\infty\int_{\mathbb{R}^n}\mathrm{e}^{-\lambda(cs+y_1)}\mathrm{e}^{-\alpha s}\int_0^s \mathrm{e}^{(\alpha-r)s_1}\Gamma(s_1,y)\,ds_1\,dyds \\
&= \int_0^\infty ds\mathrm{e}^{-(\lambda c+\alpha)s}\int_0^s ds_1\mathrm{e}^{(\alpha-r)s_1}\int_{\mathbb{R}^n}\mathrm{e}^{-\lambda y_1}\Gamma(s_1,y)\,dy \\
&= \int_0^\infty ds\mathrm{e}^{-(\lambda c+\alpha)s}\int_0^s \mathrm{e}^{(\alpha-r)s_1}\mathrm{e}^{\lambda^2 Ds_1}\,ds_1 \\
&= -\frac{1}{(\lambda^2 D - \lambda c - r)(\lambda c+\alpha)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{K}_k(c,\lambda) &= g'_\alpha(0)\mathcal{K}_{k_1}(c,\lambda) + \beta f'(0)\mathcal{K}_{k_2}(c,\lambda) \\
&= \frac{1}{\lambda c+\alpha}\left(g'_\alpha(0) - \frac{\beta f'(0)}{\lambda^2 D - \lambda c - r}\right).
\end{aligned} \tag{3.13}$$

The expression (3.13) shows that if $\lambda^{\#}(c) = \frac{c+\sqrt{c^2+4Dr}}{2D}$, then $\mathcal{K}_k(c,\lambda) < \infty$ for all $\lambda \in [0, \lambda^{\#}(c))$ and $\lim_{\lambda \nearrow \lambda^{\#}(c)} \mathcal{K}_k(c,\lambda) = \infty$ for every $c \geq 0$. In a similar way, we get

$$\tilde{F}_\alpha(w) := \int_0^\infty \int_{\mathbb{R}^n} F_\alpha(w,s,y)\, dyds = \frac{1}{\alpha}\left(g_\alpha(w) + \frac{\beta f(w)}{r}\right). \quad (3.14)$$

Note that $\mathcal{K}_k(c,0) = k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s,y)\, dyds, \forall c \geq 0$, and condition (B1) holds (i.e., $k^* > 1$) if and only if $g'(0) + \frac{\beta f'(0)}{r} > 0$. It is easy to check that conditions (B2)-(B4) hold and $\liminf_{\lambda \nearrow \lambda^{\#}(c)} \mathcal{K}_k(c,\lambda) \geq k^*$ for every $c > 0$.

We define

$$c^* := \inf\{c \geq 0 : \mathcal{K}_k(c,\lambda) < 1 \text{ for some } \lambda > 0\}.$$

According to Proposition 2.1, c^* can be uniquely determined as the positive solution of the system

$$\mathcal{K}_k(c,\lambda) = 1, \quad \frac{d}{d\lambda}\mathcal{K}_k(c,\lambda) = 0.$$

That is, (c^*, λ^*) is the unique positive solution of the system

$$\begin{cases} (g'(0) - \lambda c)(\lambda^2 D - \lambda c - r) = \beta f'(0), \\ c(\lambda^2 D - \lambda c - r)^2 = \beta f'(0)(2\lambda D - c). \end{cases} \quad (3.15)$$

Let

$$P(c,\lambda) := a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \quad (3.16)$$

where the coefficients a_i $(i = 0, \cdots, 3)$ are given in terms of the original parameters as

$$a_0 = -[\beta f'(0) + rg'(0)], \quad a_1 = -c[g'(0) - r], \quad a_2 = c^2 + Dg'(0), \quad a_3 = -cD.$$

A direct computation shows that (3.15) is equivalent to

$$P(c,\lambda)=0 \quad \text{and} \quad \frac{\partial P}{\partial \lambda}(c,\lambda)=0. \tag{3.17}$$

For any $c>0$, we have $P(c,0)<0$ and $P(c,-\infty)=+\infty$, and hence $P(c,\lambda)$ always has a real negative root. By (3.17), it follows that $P(c,\lambda)$ has two positive roots for $c>c^*$, one positive double root for $c=c^*$, and two complex roots for $0<c<c^*$.

As in [15], we now transform (3.17) so that it is expressed in terms of parameter c. Set

$$\begin{aligned} P(c,\lambda) &= P_1(c,\lambda)Q_1(c,\lambda)+R_1(c,\lambda), \\ P_1(c,\lambda) &= R_1(c,\lambda)Q_2(c,\lambda)+R_2(c), \end{aligned}$$

where $P_1(c,\lambda)=\frac{\partial P}{\partial \lambda}(c,\lambda)$, $Q_1(c,\lambda)$ and $R_1(c,\lambda)$ are the quotient and remainder of $P(c,\lambda)$ divided by $P_1(c,\lambda)$, and $Q_2(c,\lambda)$ and $R_2(c)$ are the quotient and remainder of $P_1(c,\lambda)$ divided by $R_1(c,\lambda)$, respectively. Clearly, we must have $R_2(c^*)=0$. By direct calculations, we see that $R_2(c)=0$ is equivalent to

$$18a_0a_1a_2a_3-4a_2^3a_0+a_2^2a_1^2-27a_3^2a_0^2-4a_1^3a_3=0,$$

that is,

$$\begin{aligned} \psi(c^2) := {} & 18Dc^2[c^2+Dg'(0)][g'(0)-r]a_0-4[c^2+Dg'(0)]^3a_0 \\ & +c^2[c^2+Dg'(0)]^2[g'(0)-r]^2-27D^2c^2a_0^2-4Dc^4[g'(0)-r]^3=0. \end{aligned}$$

Sorting out terms with respect to c, we have

$$\begin{aligned} \psi(c^2) = {} & c^6\{[g'(0)-r]^2-4a_0\}-4D^3g'^3(0)a_0 \\ & +c^4D\{18[g'(0)-r]a_0-12g'(0)a_0+2g'(0)[g'(0)-r]^2-4[g'(0)-r]^3\} \\ & +c^2D^2\{18g'(0)[g'(0)-r]a_0-12g'^2(0)a_0+g'^2(0)[g'(0)-r]^2-27a_0^2\}. \end{aligned}$$

Thus, $\psi({c^*}^2) = 0$. Let $\hat{c}^2$ be the largest zero of $\psi(x)$ with $\hat{c} > 0$. Clearly, $\hat{c} \geq c^*$. We further claim that $\hat{c} = c^*$. Assume, by contradiction, that $\hat{c} > c^*$. Since $\psi(\hat{c}^2) = 0$, and hence $R_2(\hat{c}) = 0$, it follows that $P(\hat{c}, \lambda)$ and $P_1(\hat{c}, \lambda)$ have the common factor $R_1(\hat{c}, \lambda)$. Thus, $P(\hat{c}, \lambda)$ has a double root, which contradicts the fact that $P(\hat{c}, \lambda)$ has three different real roots. Hence, c^* is the positive square root of the largest zero of the cubic $\psi(x)$.

The subsequent result shows that c^* is the asymptotic speed of spread for solutions of (1.2) with initial functions having compact supports. In order to obtain the convergence for $0 < c < c^*$, we need the following additional condition:

(C4) $\beta f(w) + rg(w) > 0$, $\forall w \in (0, w^*)$, and $\beta f(w) + rg(w) < 0$, $\forall w > w^*$.

We are now in a position to prove our main result in this section.

Theorem 3.1. *Let (C1)–(C3) hold and c^* be the positive square root of the largest zero of the cubic $\psi(x)$. Assume that $\phi = (\phi_1, \phi_2) \in \mathbb{X}_+$ has the property that $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, and that for every $\kappa_1 > 0$, there exists $\kappa_2 > 0$ such that $\phi_1(y) + \phi_2(y) \leq \kappa_2 e^{-\kappa_1 |y|}, \forall y \in \mathbb{R}^n$. Then the unique solution $u(t,x) = (v(t,x), w(t,x))$ of system (1.2)-(1.3) satisfies*

(i) $\lim_{t\to\infty,\, |x|\geq ct} u(t,x) = (0,0)$, $\forall c > c^*$.

(ii) If, in addition, (C4) holds, then $\lim_{t\to\infty, |x|\leq ct} u(t,x) = (v^*, w^*)$, $\forall c \in (0, c^*)$, *where w^* is the unique positive solution of $rg(w) + \beta f(w) = 0$, and $v^* = \frac{f(w^*)}{r}$.*

Proof. By Lemma 3.1, there exists a unique, bounded and non-negative solution $u(t,x) = (v(t,x), w(t,x))$ to system (1.2)-(1.3). Let $M > 0$ be a bound for $w(t,x)$, then we can choose $\alpha > \max\{g'(0), r\}$ so large that $g_\alpha(w) = \alpha w + g(w), \forall w \in [0, M]$, and hence, $g_\alpha(w(t,x)) = \alpha w(t,x) + g(w(t,x)), \forall t \geq 0, x \in \mathbb{R}^n$. Then $w(t,x)$ is a solution of (3.9).

Note that $\Gamma(t,\cdot) > 0, \forall t > 0$, it follows from (3.10) and the assumption on $\phi = (\phi_1, \phi_2)$ that $w_0(t,\cdot) > 0$ for $t > 0$. Since $\int_{\mathbb{R}^n} \Gamma(t, x - y)dy = 1, \forall t \geq 0, x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
w_0(t,x) &= \mathrm{e}^{-\alpha t}\phi_2(x) + \beta \int_0^t ds \mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{-rs} \int_{\mathbb{R}^n} \Gamma(s, x-y)\phi_1(y)\, dy \\
&\leq \mathrm{e}^{-\alpha t}\phi_2(x) + \beta M_1 \int_0^t e^{-\alpha t + (\alpha - r)s} ds,
\end{aligned}
$$

where $M_1 = \sup\limits_{y \in \mathbb{R}^n} \phi_1(y)$. Therefore, $\lim\limits_{t\to\infty} w_0(t,x) = 0$ uniformly in $x \in \mathbb{R}^n$. Note that $v(t,x)$ is determined from (3.3) by $w(t,x)$. In the case where (C4) holds, it follows from (3.14) that $\tilde{F}_\alpha(w) > w, \forall w \in (0, w^*)$ and $\tilde{F}_\alpha(w) < w, \forall w \in (w^*, \bar{w}]$, where $\bar{w} = \min\{w^* + 1, M\}$.

We spend the rest of this proof on checking that $w_0(t,x)$ is admissible. Indeed, given $c, \lambda > 0$ with $\mathcal{K}_k(c,\lambda) < 1$, $\mathcal{K}_{k_1}(c,\lambda)$ and $\mathcal{K}_{k_2}(c,\lambda)$ are finite numbers. Therefore, $\lambda^2 D - \lambda c - r < 0$. Note that for every $w \in \mathbb{R}^n$ with $|w| = 1$, $-|y| \leq w \cdot y \leq |y|, \forall y \in \mathbb{R}^n$, where $\cdot$ is the inner product on $\mathbb{R}^n$. By the assumption on ϕ_1 and ϕ_2, there exists $\gamma > 0$ such that $\phi_i(y) \leq \gamma e^{-\lambda|y|} \leq \gamma e^{\lambda w \cdot y}, \forall y \in \mathbb{R}^n, i = 1, 2$. Note that

$$
\int_{\mathbb{R}^n} \Gamma(t,y) e^{-\lambda w \cdot y} dy = \int_{\mathbb{R}^n} \Gamma(t,y) e^{-\lambda y_1} dy = e^{\lambda^2 D t}.
$$

We then have

$$\begin{aligned}
w_0(t,x) &\le \mathrm{e}^{-\alpha t}\gamma e^{-\lambda|x|} + \beta\int_0^t ds\mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{-rs}\int_{\mathbb{R}^n}\Gamma(s,x-y)\gamma e^{\lambda w\cdot y}\,dy \\
&= \gamma\mathrm{e}^{-\alpha t-\lambda|x|} + \beta\gamma\int_0^t ds\mathrm{e}^{-\alpha(t-s)}\mathrm{e}^{-rs}\int_{\mathbb{R}^n}\Gamma(s,y)e^{\lambda w\cdot(x-y)}\,dy \\
&= \gamma\mathrm{e}^{-\alpha t-\lambda|x|} + \beta\gamma e^{-\alpha t+\lambda w\cdot x}\int_0^t ds\mathrm{e}^{(\alpha-r)s}\int_{\mathbb{R}^n}\Gamma(s,y)e^{-\lambda w\cdot y}\,dy \\
&= \gamma\mathrm{e}^{-\alpha t-\lambda|x|} + \beta\gamma e^{-\alpha t+\lambda w\cdot x}\int_0^t \mathrm{e}^{[\lambda^2 D+(\alpha-r)]s}\,ds \\
&= \gamma\mathrm{e}^{-\alpha t-\lambda|x|} + \frac{\beta\gamma e^{-\alpha t+\lambda w\cdot x}}{\lambda^2 D+(\alpha-r)}\left[\mathrm{e}^{[\lambda^2 D+(\alpha-r)]t}-1\right] \\
&\le \gamma\mathrm{e}^{-\alpha t-\lambda|x|} + \frac{\beta\gamma}{\lambda^2 D+(\alpha-r)}e^{(\lambda^2 D-r)t+\lambda w\cdot x}.
\end{aligned}$$

Letting $w = -\frac{x}{|x|}$, and using the inequality $\lambda^2 D - r < \lambda c$, we obtain

$$w_0(t,x) \le \gamma\mathrm{e}^{\lambda(ct-|x|)} + \frac{\beta\gamma}{\lambda^2 D+(\alpha-r)}e^{\lambda(ct-|x|)}, \quad \forall t\ge 0, x\in\mathbb{R}^n.$$

Therefore, $w_0(t,x)$ is admissible. Consequently, the results for $w(t,x)$ follow from Remark 2.1 and Theorems 2.1 and 2.2. By (3.3), it then follows that the corresponding results hold for $v(t,x)$. The proof is complete.

As an application, let us consider system (1.1), where $D,\mu,\gamma_1,\gamma_2,r$ and K are positive constants. It is easy to verify that system (1.1) satisfies conditions (C1)–(C4) provided $r > \frac{\mu\gamma_1}{\mu+\gamma_2}$. Setting

$$\begin{aligned}\psi_0(x) := {} & x^3[(r-\gamma_1-\mu-\gamma_2)^2+4(\mu r+r\gamma_2-\mu\gamma_1)] \\ & + 4D^3(r-\gamma_1)^3(\mu r+r\gamma_2-\mu\gamma_1) \\ & + x^2D[-18(r-\gamma_1-\mu-\gamma_2)(\mu r+r\gamma_2-\mu\gamma_1) \\ & + 12(r-\gamma_1)(\mu r+r\gamma_2-\mu\gamma_1) \\ & + 2(r-\gamma_1)(r-\gamma_1-\mu-\gamma_2)^2-4(r-\gamma_1-\mu-\gamma_2)^3] \\ & + xD^2[-18(r-\gamma_1)(r-\gamma_1-\mu-\gamma_2)(\mu r+r\gamma_2-\mu\gamma_1) \\ & + 12(r-\gamma_1)^2(\mu r+r\gamma_2-\mu\gamma_1) \\ & + (r-\gamma_1)^2(r-\gamma_1-\mu-\gamma_2)^2-27(\mu r+r\gamma_2-\mu\gamma_1)^2],\end{aligned}$$

we then have the following result.

Proposition 3.1. *Let $r > \frac{\mu\gamma_1}{\mu+\gamma_2}$ hold, and c^* be the positive square root of the largest zero of the cubic $\psi_0(x)$. Assume that $\phi = (\phi_1, \phi_2) \in \mathbb{X}_+$ has the property that $\phi_1(\cdot)+\phi_2(\cdot) \not\equiv 0$, and that for every $\kappa_1 > 0$, there exists $\kappa_2 > 0$ such that $\phi_1(y)+\phi_2(y) \le \kappa_2 e^{-\kappa_1|y|}, \forall y \in \mathbb{R}^n$. Then the unique solution $u(t,x) = (v(t,x), w(t,x))$ of system (1.1) with (1.3) satisfies*

(i) $\lim_{t\to\infty,\,|x|\ge ct} u(t,x) = (0,0),\ \forall c > c^*$.

(ii) $\lim_{t\to\infty,|x|\le ct} u(t,x) = (v^*, w^*)$, *$\forall c \in (0, c^*)$, where* $w^* = K\left(1-\frac{\mu\gamma_1}{r(\mu+\gamma_2)}\right)$ *and* $v^* = \frac{\gamma_1 K}{\mu+\gamma_2}\left(1-\frac{\mu\gamma_1}{r(\mu+\gamma_2)}\right)$.

Remark 3.1. (3.17) implies that the spreading speed c^* of (1.2) can be obtained as the largest value c such that the polynomial $P(c,\lambda)$ defined by (3.16) has a real positive double root. For system (1.1), c^* defined in Proposition 3.1 coincides with the spreading rate $\bar{c}$ in [3, Theorem 1].

4 Traveling wave solutions

In this section, we consider the existence and nonexistence of traveling wave solutions of system (1.2) with $n = 1$. We will show that there is a minimal wave speed for monotone traveling waves and it coincides with the spreading speed c^* obtained in section 3.

Recall that a solution $(v(t,x), w(t,x))$ of (1.2) is said to be a traveling wave solution if it is of the form $(v(t,x), w(t,x)) = (U_1(x + ct), U_2(x + ct))$. The parameter c is called the wave speed, and the function $(U_1(\cdot), U_2(\cdot))$ is called the wave profile. We will impose the following conditions on the wave profile:

$$U_i(\cdot) \text{ is positive and bounded on } \mathbb{R}, \text{ and } \lim_{\xi\to-\infty} U_i(\xi) = 0,\ i = 1, 2. \tag{4.1}$$

Theorem 4.1. *Let (C1)–(C3) hold, and let c^*, v^*, w^* be defined as in Theorem 3.1. Then the following statements are valid:*

(i) System (1.2) with $n = 1$ subject to (4.1) admits no traveling wave solution with wave speed $c \in (0, c^*)$.

(ii) Assume in addition that (C4) holds, $f''(0)$ exists, and there exist $\delta, b, \theta > 0$ such that $g'(u) - g'(0) \geq -bu^\theta, \forall u \in [0, \delta]$. Then for every $c \geq c^*$, system (1.2) with $n = 1$ has a monotone traveling wave connecting $(0, 0)$ and (v^*, w^*) with speed c.

Proof. (i) Assume that $(v(t,x), w(t,x)) = (U_1(x+ct), U_2(x+ct))$ is a traveling wave solution of (1.2) with $n = 1$ subject to (4.1). Then $(v(t,x), w(t,x))$ is bounded, and we can choose $\alpha > 0$ so large that $g_\alpha(w(t,x)) = \alpha w(t,x) + g(w(t,x)), \forall t \geq 0, x \in \mathbb{R}^n$. A similar argument as in Section 3 shows that $(v(t,\cdot), w(t,\cdot))$ satisfies the integral equations

$$v(t) = T(t-r)v(r) + \int_r^t T(t-s)f(w(s))\,ds, \tag{4.2}$$

$$w(t) = e^{-\alpha(t-r)}w(r) + \int_r^t e^{-\alpha(t-s)}[g_\alpha(w(s)) + \beta v(s)]\,ds, \quad \forall t \geq r,\ r \in \mathbb{R}, \tag{4.3}$$

where $T(t)$ is defined as in (3.2). By [10, Proposition 4.3], we have

$$v(t,x) = \int_0^\infty ds\mathrm{e}^{-rs} \int_{\mathbb{R}^n} \Gamma(s,y)f(w(t-s,x-y))\,dy = U_1(x+ct) \tag{4.4}$$

and

$$\begin{aligned}
w(t,x) &= \int_0^\infty \mathrm{e}^{-\alpha s}[g_\alpha(w(t-s,x)) + \beta v(t-s,x)]\,ds \\
&= \int_0^\infty \mathrm{e}^{-\alpha s}\Bigg[g_\alpha(w(t-s,x)) \\
&\quad + \beta \int_{\mathbb{R}} dy \int_0^\infty \mathrm{e}^{-rs_1}\Gamma(s_1,y)f(w(t-s-s_1,x-y))\,ds_1\Bigg]\,ds \\
&= \int_0^\infty g_\alpha(w(t-s,x))\mathrm{e}^{-\alpha s}\,ds \\
&\quad + \beta \int_0^\infty ds \int_{\mathbb{R}} dy \int_0^\infty \mathrm{e}^{(\alpha-r)s_1}\Gamma(s_1,y)f(w(t-s-s_1,x-y))\mathrm{e}^{-\alpha(s+s_1)}\,ds_1 \\
&= \int_0^\infty g_\alpha(w(t-s,x))\mathrm{e}^{-\alpha s}\,ds \\
&\quad + \beta \int_{\mathbb{R}} dy \int_0^\infty ds_1 \int_{s_1}^\infty \mathrm{e}^{(\alpha-r)s_1}\Gamma(s_1,y)f(w(t-s,x-y))\mathrm{e}^{-\alpha s}\,ds \\
&= \int_0^\infty \int_{\mathbb{R}} g_\alpha(w(t-s,x-y))\mathrm{e}^{-\alpha s}\delta(y)\,dyds \\
&\quad + \beta \int_0^\infty \int_{\mathbb{R}} f(w(t-s,x-y))\mathrm{e}^{-\alpha s} \int_0^s \mathrm{e}^{(\alpha-r)s_1}\Gamma(s_1,y)\,ds_1\,dyds \\
&= \int_0^\infty \int_{\mathbb{R}} F_\alpha(w(t-s,x-y),s,y)\,dyds,
\end{aligned} \tag{4.5}$$

where $F_\alpha(w,s,y)$ is defined as in (3.11). It then follows that $w(t,x) = U_2(x+ct)$ is a traveling wave solution of system (4.5). However, (C1)–(C3) imply that conditions (A) and (B) hold for (4.5). Then the assumptions of Remark 2.1 and Theorem 2.4 are satisfied and such a solution does not exist.

(ii) We choose $\alpha > 0$ so large that $g_\alpha(w) = \alpha w + g(w), \forall w \in [0, w^*]$. Since g_α and f are nondecreasing, so is $F_\alpha(\,\cdot\,, s, y)$ in (3.11). Assume that

$(v(t,x), w(t,x)) = (U_1(x+ct), U_2(x+ct))$ is a monotone traveling wave connecting $(0,0)$ and (v^*, w^*) with speed $c \geq c^*$ of

$$v(t,x) = \int_0^\infty ds e^{-rs} \int_{\mathbb{R}} \Gamma(s,y) f(w(t-s, x-y))\, dy, \quad (4.6)$$

$$w(t,x) = \int_0^\infty \int_{\mathbb{R}} F_\alpha(w(t-s, x-y), s, y)\, dy ds, \quad (4.7)$$

where $F_\alpha(w, s, y)$ is given in (3.11). Since the process in (4.5) is invertible, by [10, Proposition 4.3] we see that $(v(t,x), w(t,x))$ satisfies (4.2)-(4.3). It then follows that $(v(t,x), w(t,x))$ solves (1.2). Thus, $(v(t,x), w(t,x)) = (U_1(x+ct), U_2(x+ct))$ is a monotone traveling wave connecting $(0,0)$ and (v^*, w^*) with speed c of system (1.2) with $n = 1$ subject to (4.1). Therefore, it suffices to prove the existence of monotone traveling solution of (4.6)-(4.7).

Since $f''(0)$ exists, we can find two numbers $\delta_0 > 0$ and $a > 0$ such that $f(u) \geq f'(0)w - aw^2, \forall w \in [0, \delta_0]$. Furthermore, if $\delta > 0$ is chosen small enough, we have

$$\begin{aligned} g_\alpha(w) &= \alpha w + g(w) = w\left(\alpha + \int_0^1 g'(\xi w)\, d\xi\right) \\ &= w\left(\alpha + g'(0) + \int_0^1 [g'(\xi w) - g'(0)]\, d\xi\right) \\ &\geq w\left(g_\alpha'(0) - \int_0^1 b(\xi w)^\theta\, d\xi\right) \\ &= w\left(g_\alpha'(0) - \frac{b}{\theta+1} w^\theta\right), \quad \forall u \in [0, \delta]. \end{aligned}$$

Without loss of generality, we can assume that $\theta \in (0,1]$. Then there exist $\delta_1 > 0$ and $b_1 > 0$ such that

$$F_\alpha(w,s,y) \geq \left(w - b_1 w^{1+\theta}\right) k(s,y), \quad \forall w \in [0,\delta_1],\ (s,y) \in \mathbb{R}_+ \times \mathbb{R}.$$

Applying Remark 2.1 and Theorem 2.3 to (4.7), it follows that for each $c > c^*$, (4.7) admits a monotone traveling wave $w(t,x) = U_2(x+ct)$ connecting 0 and w^*. Define $v(t,x)$ as in (4.6), we then have

$$v(t,x) = \int_0^\infty ds e^{-rs} \int_{\mathbb{R}} \Gamma(s,y) f(U_2(x-y+c(t-s)))\, dy = U_1(x+ct),$$

where $U_1(\xi) = \int_0^\infty ds e^{-rs} \int_{\mathbb{R}} \Gamma(s,y) f(U_2(\xi - y - cs))\, dy$. Obviously, $U_1'(\xi) > 0$. By the dominant convergence theorem, $\lim_{\xi\to-\infty} U_1(\xi) = 0$, and $\lim_{\xi\to\infty} U_1(\xi) = \frac{f(w^*)}{r} = v^*$. Therefore, $(v(t,x), w(t,x))$ is a monotone traveling wave connecting $(0,0)$ and (v^*,w^*) with speed $c > c^*$ of (4.6)-(4.7).

By the limiting arguments (see, e.g., [15]), as applied to the ordinary differential system of the wave profile resulting from (1.2), we can prove the existence of monotone traveling wave of (1.2) with speed c^*. The proof is complete.

Returning to system (1.1), we have the following result.

Proposition 4.1. *Let $r > \frac{\mu\gamma_1}{\mu+\gamma_2}$ hold, and let c^*, v^*, w^* be defined as in Proposition 3.1. Then the following statements are valid:*

(i) System (1.1) with $n = 1$ subject to (4.1) admits no traveling wave solution with wave speed $c \in (0,c^*)$.

(ii) For every $c \geq c^*$, system (1.1) with $n = 1$ has a monotone traveling wave connecting $(0,0)$ and (v^*,w^*) with speed c.

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