Global Convergence in Monotone and Uniformly Stable Recurrent Skew-Product Semiflows

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Dedicated to George Sell on the occasion of his 70th birthday

Abstract The 1-covering property of omega limit sets is established for monotone and uniformly stable skew-product semiflows with a minimal base flow. Then the convergence result for monotone and subhomogeneous semiflows is applied to obtain the asymptotic recurrence of solutions to a linear recurrent nonhomogeneous ordinary differential system and a nonlinear recurrent reaction-diffusion equation.

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1 Introduction

It is well known that the skew-product semiflows approach is a powerful tool in the study of linear and nonlinear nonautonomous evolution systems (see, e.g., [4, 5, 8, 9]). This approach has also been used extensively to generalize certain important results on global dynamics of monotone autonomous and periodic systems (see, e.g., [1, 12]) to monotone almost periodic and recurrent systems, see, e.g., [2, 6, 10, 11, 14] and references therein. Recently, Jiang and Zhao [2] established the 1-covering property of omega limit sets for monotone and uniformly stable

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skew-product semiflows with a minimal and distal base flow, and obtained the asymptotic almost periodicity of bounded solutions for monotone almost periodic systems with a first integral or subhomogeneous nonlinearity. It is natural to expect that there are similar convergence results in a more general nonautonomous case. To be more precise, we take into account a nonautonomous ordinary differential system $x' = f(t,x)$ with $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and let $H(f)$ be the hull of f with respect to the compact open topology. The function f is said to be time recurrent if $H(f)$ is compact and the translation flow is minimal on $H(f)$. Note that any uniformly almost automorphic function in $C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ is time recurrent (see [10, Sect. I.3.1]). Accordingly, one may ask under what conditions on f, every bounded solution of such a recurrent system is asymptotic to a recurrent full solution. Clearly, one can not use the abstract results in [2] to prove this kind of asymptotic recurrence since the base flow is only minimal in our current case. More recently, Novo, Obaya and Sanz [6] have presented some general results on the structure of uniformly stable and uniformly asymptotically stable compact invariant sets for skew-product semiflows with a minimal base flow. In particular, they proved that the omega limit set of a precompact and uniformly stable forward orbit admits a minimal and fiber distal flow extension. Further, we observe that if a compact invariant set K of such a skew-product semiflow admits a flow extension and its section map is continuous, then K is an 1-covering of the base space whenever its intersection with some fiber is a singleton (see Lemma 2.4).

The purpose of our current paper is to generalize two abstract convergence results in [2] to skew-product semiflows with a minimal base flow in such a way that we can obtain the asymptotic recurrence for monotone and subhomogeneous or uniformly stable recurrent evolution systems. For a monotone and subhomogeneous semiflow, we can not directly utilize the afore-mentioned results under the norm-induced metric. However, we can prove the 1-covering property by employing the uniform stability of compact and strongly positive invariant sets with respect to the part metric (Lemma 3.1). For a monotone and uniformly stable skew-product semiflow, we will introduce a new assumption of the strong componentwise separating property (see (A4)) to prove the convergence result without assuming that the positive cone P is solid (i.e., $Int(P) \neq \emptyset$). We should point out that our arguments were highly motivated by those in [2, 6, 14].

The remained part of this paper is organized as follows. In Sect. 2, we present some basic concepts and results in the theory of skew-product semiflows. In Sect. 3, we prove the 1-covering property of omega limit sets for subhomogeneous and strongly monotone skew-product semiflows in the case where $Int(P) \neq \emptyset$ (Theorem 3.3), and for monotone and uniformly stable skew-product semiflows with the strong componentwise separating property without assuming $Int(P) \neq \emptyset$ (Theorem 3.4). Section 4 is devoted to the application of the convergence result to a linear recurrent nonhomogeneous ordinary differential system (Theorem 4.1) and a nonlinear recurrent reaction-diffusion equation (Theorem 4.2).

2 Preliminaries

Let $\mathbb{R}_+ = [0,\infty)$, (M,d_M) be a complete metric space, and (Y,d_Y) be a compact metric space. We use d to denote the metric on the product space $M \times Y$, which is induced by d_M and d_Y.

A continuous flow $\sigma : Y \times \mathbb{R} \to Y$ is said to be minimal if Y contains no nonempty, proper, closed invariant subset. Clearly, a flow $\sigma : Y \times \mathbb{R} \to Y$ is minimal if and only if every full orbit is dense in Y.

Consider a continuous skew-product semiflow $\Pi : M \times Y \times \mathbb{R}_+ \to M \times Y$ defined by

$$\Pi(x,y,t) = (u(x,y,t), \sigma(y,t)), \quad \forall (x,y,t) \in M \times Y \times \mathbb{R}_+.$$

For convenience, we also use Π_t to denote $\Pi(\cdot,t)$. A subset K of $M \times Y$ is said to be Π-invariant if $\Pi_t(K) = K$ for all $t \geq 0$. Further, the skew-product semiflow Π is said to be recurrent if its base flow $(Y,\sigma,\mathbb{R})$ is minimal. Throughout this paper, we always assume that Π is recurrent.

Definition 2.1. Let Π be a skew-product semiflow on $M \times Y$. A forward orbit $\Pi_t(x_0,y_0)$, $t \geq 0$, is said to be uniformly stable if for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, called the modulus of uniform stability, such that

$$d(\Pi(x,y_0,\tau+t), \Pi(x_0,y_0,\tau+t)) < \varepsilon, \quad \forall t \geq 0$$

whenever $\tau \geq 0$ and $d_M(u(x,y_0,\tau), u(x_0,y_0,\tau)) < \delta$. Let $K \subset M \times Y$ be a compact Π-invariant set. The semiflow $(K,\Pi,\mathbb{R}_+)$ is uniformly stable if for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $d(\Pi(x_1,y,t), \Pi(x_2,y,t)) < \varepsilon$, $\forall t \geq 0$, whenever $(x_1,y),(x_2,y) \in K$ with $d_M(x_1,x_2) < \delta$.

A compact Π-invariant set $K \subset M \times Y$ which admits a flow extension is said to be fiber distal if for any $y \in Y$, any two distinct points (x_1,y), $(x_2,y) \in K$, we have

$$\inf_{t\in\mathbb{R}} d_M(u(x_1,y,t), u(x_2,y,t)) > 0.$$

For any given set $K \subset M \times Y$, we define $K_y := \{x \in M : (x,y) \in K\}$, $\forall y \in Y$, and

$$Y_0(K) := \{\tilde{y} \in Y : \text{for any } \tilde{x} \in K_{\tilde{y}}, y \in Y \text{ and any sequence } \{t_i\} \subset \mathbb{R} \text{ with } \sigma_{t_i}(y) \to \tilde{y}, \text{ there is a sequence } \{x_i\} \subset K_y \text{ such that } u(x_i,y,t_i) \to \tilde{x}\}.$$

Further, K is said to be an 1-covering of Y if K_y is a singleton for all $y \in Y$.

Theorem 2.2 [6, Theorem 3.4]. *Let $K \subset M \times Y$ be a compact Π-invariant set such that every point of K admits a backward orbit. If the semiflow $(K,\Pi,\mathbb{R}_+)$ is uniformly stable, then it admits a flow extension which is fiber distal and uniformly stable as $t \to -\infty$. Further, the section map for K, $y \in Y \mapsto K_y$, is continuous at every $y \in Y$ with respect to the Hausdorff metric.*

Theorem 2.3 [6, Proposition 3.6]. *Let $\{\Pi(\tilde{x},\tilde{y},t) : t \geq 0\}$ be a precompact forward orbit of the skew-product semiflow Π and let $\tilde{K}$ denote the omega-limit set of $(\tilde{x},\tilde{y})$. Then the following statements are valid:*

(1) If $\tilde{K}$ contains a minimal set K which is uniformly stable, then $\tilde{K} = K$ and it admits a fiber distal flow extension.
(2) If the forward orbit is uniformly stable, then its omega-limit set $\tilde{K}$ is a uniformly stable minimal set which admits a fiber distal flow extension.

Lemma 2.4. *Let K be a compact Π-invariant set such that it admits a flow extension. If the section map for K is continuous at every $y \in Y$ with respect to the Hausdorff metric, then $Y_0(K) = Y$, and K is an 1-covering of Y whenever K_{y_0} is a singleton for some $y_0 \in Y$.*

Proof. Clearly, it suffices to prove that $Y \subseteq Y_0(K)$. Let $\tilde{y} \in Y$ be given. For any $\tilde{x} \in K_{\tilde{y}}$, $y \in Y$ and any sequence $\{t_i\} \subset \mathbb{R}$ with $\sigma_{t_i}(y) \to \tilde{y}$, by the continuity of the section map $y \in Y \mapsto K_y$, we can deduce that there is a sequence $x_i' \in K_{\sigma_{t_i}(y)}$ such that $x_i' \to \tilde{x}$. Note that K is invariant under the flow extension Π, we have $K_{\sigma_{t_i}(y)} = u(K_y, y, t_i)$. It then follows that there is a sequence $\{x_i\} \subseteq K_y$ such that $x_i' = u(x_i, y, t_i)$ and $u(x_i, y, t_i) \to \tilde{x}$, and hence, $\tilde{y} \in Y_0(K)$. This proves that $Y_0(K) = Y$. Now we assume that $K_{y_0} = \{x_0\}$. For any given $y \in Y$, the minimality of the base flow implies that there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that $\sigma_{t_n}(y_0) \to y$ as $n \to \infty$. Since $y \in Y = Y_0(K)$, it follows from the definition of $Y_0(K)$ that for any $\tilde{x} \in K_y$, there holds $\lim_{n\to\infty} u(x_0, y_0, t_n) = \tilde{x}$. This implies that K_y is a singleton. □

Let (X,P) be an ordered Banach space. For $x_1, x_2 \in X$, we write $x_1 \leq x_2$ if $x_2 - x_1 \in P$; $x_1 < x_2$ if $x_2 - x_1 \in P \setminus \{0\}$; $x_1 \ll x_2$ if $Int(P) \neq \emptyset$ and $x_2 - x_1 \in Int(P)$. A subset U of X is said to be order convex if for any $a, b \in U$ with $a < b$, the order interval $[a,b]_X := \{x \in X : a \leq x \leq b\}$ is contained in U.

In the rest of this paper, we assume that V is a closed and order convex subset of the positive cone P.

Let $Q : V \times Y \to Y$ be the natural projection. For a skew-product semiflow, we always use the order relations on each fiber $Q^{-1}(y)$. We write $(x_1,y) \geq_y (>_y, \gg_y)(x_2,y)$ if $x_1 \geq x_2 (x_1 > x_2, x_1 \gg x_2)$. Without any confusion, we will drop the subscript "y".

A skew-product semiflow Π_t on $V \times Y$ is said to be monotone (strongly monotone) if

$$\Pi_t(x_1,y) \leq (\ll) \Pi_t(x_2,y)$$

whenever $t > 0$ and $(x_1,y) \leq (x_2,y)((x_1,y) < (x_2,y))$.

To study omega limit sets of a monotone skew-product semiflow Π, we need the following assumptions.

(A1) Every compact subset in V has both the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.).
(A2) For every $(x,y) \in V \times Y$, there is a $t_0 = t_0(x,y)$ such that $\{\Pi_t(x,y) : t \geq t_0\}$ is precompact.

(A3) The skew-product semiflow $\Pi_t : V \times Y \to V \times Y$ is monotone, and every forward orbit of Π_t is uniformly stable.

Note that if (A2) holds, then every omega limit set $\omega(x,y)$, $(x,y) \in V \times Y$, is a nonempty, compact and Π-invariant subset in $V \times Y$.

Let $K \subset V \times Y$ be a compact Π-invariant set. For any given $y \in Y$, we define

$$(p(y),y) = g.l.b. \quad \text{of } K \cap Q^{-1}(y)$$

and

$$(q(y),y) = l.u.b. \quad \text{of } K \cap Q^{-1}(y).$$

By (A1), $p(y)$ and $q(y)$ are well defined.

In view of Theorems 2.2 and 2.3 and Lemma 2.4, it is easy to see that the proof of [2, Propositions 3.1 and 3.2] still works in the case where the base flow is only minimal. Thus, we have the following result.

Theorem 2.5 [2, Propositions 3.1 and 3.2]. *Assume that (A1) − (A3) hold. Let $K := \omega(x_0, y_0)$, $(x_0, y_0) \in V \times Y$, be fixed. Then the following statements are valid:*

(1) For any $y \in Y$, both $\omega(p(y),y)$ and $\omega(q(y),y)$ are 1-coverings of Y.
(2) Let $(p_*(y),y) := \omega(p(y),y) \cap Q^{-1}(y)$ and $(q_*(y),y) := \omega(q(y),y) \cap Q^{-1}(y)$. Then $(p_*(y),y) \leq (p(y),y) \leq (z,y) \leq (q(y),y) \leq (q_*(y),y)$, $\forall (z,y) \in K$.
(3) For any $y \in Y$ and $t \in \mathbb{R}$, there holds $u(p_*(y),y,t) = p_*(\sigma_t(y))$.

3 Global Convergence

In this section, we establish the 1-covering property for omega limit sets of monotone and uniformly stable recurrent skew-product semiflows.

We first consider a monotone skew-product semiflow $(\Pi, V \times Y, \mathbb{R}_+)$ in the case where $Int(P) \neq \emptyset$. We use the notations

$$\gamma^+(x,y) := \{\Pi_t(x,y) : t \geq 0\}$$

and

$$\gamma(x,y) := \{\Pi_t(x,y) : t \in \mathbb{R}\}$$

to denote the forward orbit and full orbit (if it exists) through (x,y), respectively. Π is said to be subhomogeneous if

$$u(\alpha x, y, t) \geq \alpha u(x,y,t), \quad \forall (x,y) \in V \times Y, \alpha \in (0,1), t \geq 0. \tag{1}$$

As in [2, 7, 13, 14], we define the part metric ρ on $Int(P)$ by

$$\rho(x_1,x_2) := \inf\{\ln \alpha : \alpha \geq 1 \text{ and } \alpha^{-1} x_1 \leq x_2 \leq \alpha x_1\}, \quad \forall x_1, x_2 \in Int(P).$$

Then $(Int(P),\rho)$ is a metric space (see, e.g., [7], [13]). We denote the metrics of the product space $Int(P)\times Y$ induced by (ρ,d_Y) and $(\|\cdot\|,d_Y)$ as d_ρ and $d_{\|\cdot\|}$, respectively.

Lemma 3.1 [14, Claim 1]. *If $(\Pi,V\times Y,\mathbb{R}_+)$ is monotone and subhomogeneous, then*

$$\rho(u(x_1,y,t),u(x_2,y,t))\le\rho(x_1,x_2)$$

for all $x_1,x_2\in Int(P)$ and $(y,t)\in Y\times\mathbb{R}_+$ with $u(x_i,y,t)\in Int(P), i=1,2$.

Clearly, Lemma 3.1 implies that for any two forward orbits $\gamma^+(x_i,y)\subset Int(P)\times Y, i=1,2$, we have

$$d_\rho(\Pi_{t+\tau}(x_1,y),\Pi_{t+\tau}(x_2,y))\le d_\rho(\Pi_\tau(x_1,y),\Pi_\tau(x_2,y)),\quad\forall t\ge0,\ \tau\ge0. \tag{2}$$

Given $x_0\in Int(P)$, we can choose a real number $r>0$ such that the closed norm ball $\overline{B}(x_0,2r):=\{x\in X:\|x-x_0\|\le2r\}\subset Int(P)$. Then for any $x\in\overline{B}(x_0,r)$, there holds $\overline{B}(x,r)\subset Int(P)$. By [3, Lemma 2.3 (i)], we have

$$\rho(x,x_0)\le\ln\left(1+\frac{\|x-x_0\|}{r}\right),\quad\forall x\in\overline{B}(x_0,r). \tag{3}$$

Lemma 3.2 [2, Lemma 5.1]. *Assume that $\gamma^+(x_0,y_0)\subset Int(P)\times Y$ is precompact in $(Int(P)\times Y,d_{\|\cdot\|})$ and its omega limit set $\omega(x_0,y_0)_{d_{\|\cdot\|}}\subset Int(P)\times Y$. Then $\gamma^+(x_0,y_0)$ is also precompact in $(Int(P)\times Y,d_\rho)$, and $\omega(x_0,y_0)_{d_\rho}=\omega(x_0,y_0)_{d_{\|\cdot\|}}$.*

Theorem 3.3. *Assume that the skew-product semiflow $(\Pi,V\times Y,\mathbb{R}_+)$ is subhomogeneous and strongly monotone, and (A2) holds. If Π admits a forward orbit $\gamma^+(x_0,y_0)\subset Int(P)\times Y$ such that $\omega(x_0,y_0)\subset Int(P)\times Y$, then for any $(x,y)\in V\times Y$ with $x\gg0$, $\omega(x,y)$ is an 1-covering of Y, and $\lim_{t\to\infty}\|u(x,y,t)-u(x^*,y,t)\|=0$, where $(x^*,y)=\omega(x,y)\cap Q^{-1}(y)$.*

Proof. For any $(x,y)\in V\times Y$ with $x\gg0$, we can choose a point $(\bar{x},y)\in\omega(x_0,y_0)\subset Int(P)\times Y$ such that $x\gg\alpha\bar{x}$ for some sufficiently small $\alpha\in(0,1)$. It then follows that

$$\Pi_t(x,y)=(u(x,y,t),\sigma_t(y))\ge(u(\alpha\bar{x},y,t),\sigma_t(y))\ge(\alpha u(\bar{x},y,t),\sigma_t(y)),\quad\forall t\ge0,$$

and hence $K:=\omega(x,y)\subset Int(P)\times Y$. Note that $\omega(x,y)=\omega(\tilde{x},\tilde{y})$ for any $(\tilde{x},\tilde{y})\in\gamma^+(x,y)$. Without loss of generality, we then assume that $\gamma^+(x,y)\subset Int(P)\times Y$. Clearly, Lemma 3.2 implies that $K=\omega(x,y)_{d_\rho}$. In view of Lemma 3.1, (2) and (3), it is easy to check that Π is a semiflow on the compact metric space (K,d_ρ), and that any Π-invariant subset of K is uniformly stable in d_ρ. It then follows from Theorems 2.2 and 2.3 that K is a minimal set which admits a fiber distal flow extension, and the section map for K, $y\in Y\mapsto K_y$, is continuous at every $y\in Y$ with respect to d_ρ. Let $Y_0(K)$ be defined with respect to d_ρ. Thus, Lemma 2.4 implies that $Y_0:=Y_0(K)=Y$ in our current case. By [10, Theorem II.3.1], we then conclude that for any $y\in Y$, $K\cap Q^{-1}(y)$ contains no pair of strongly ordered distinct points. By the same contradiction argument as in [2, Theorem 5.1], it follows that $Card(K\cap Q^{-1}(y))=1$ for all $y\in Y$, that is, $\omega(x,y)$ is an 1-covering of Y.

To prove $\lim_{t\to\infty}\|u(x,y,t)-u(x^*,y,t)\|=0$, we assume, by contradiction, that there exist an $\varepsilon_0>0$ and a sequence $t_n\to\infty$ such that $\|u(x,y,t_n)-u(x^*,y,t_n)\|\ge\varepsilon_0$, $\forall n\ge1$. Clearly, $\omega(x^*,y)\subseteq K=\omega(x,y)$. In view of (A2), we can further assume, without loss of generality, that $\lim_{n\to\infty}\Pi(x,y,t_n)=(x_1^*,y^*)\in K$ and $\lim_{n\to\infty}\Pi(x^*,y,t_n)=(x_2^*,y^*)\in K$. Since $Card(K\cap Q^{-1}(y^*))=1$, we have $x_1^*=x_2^*$. Thus, $0=\|x_1^*-x_2^*\|=\lim_{n\to\infty}\|u(x,y,t_n)-u(x^*,y,t_n)\|\ge\varepsilon_0$, a contradiction. □

Next, we establish the 1-covering property of omega limit sets for a monotone and uniformly stable skew-product semiflow with the strong componentwise separating property without assuming $Int(P)\ne\emptyset$.

Let (X_i,P_i), $1\le i\le n$, be ordered Banach spaces. For each $I=\{j_1,j_2,\cdots,j_m\}\subset N:=\{1,2,\cdots,n\}$, we define

$$X_I:=\prod_{k=1}^m X_{j_k},\quad P_I:=\prod_{k=1}^m P_{j_k}.$$

Then (X_I,P_I) is an ordered Banach space. Let $\le_I$ and $<_I$ be the orders induced by P_I in X_I. In the case where $I=N$, we use (X,P) to denote the ordered Banach space (X_N,P_N), and omit the order subscripts to get the orders $\le$ and $<$ in X, respectively. For each $1\le i\le n$, let $Q_i:X\times Y\mapsto X_i$ be the projection mapping defined by $Q_i(x,y)=x_i$.

Let V be a closed and order convex subset of X. For the skew-product semiflow

$$\Pi:V\times Y\times\mathbb{R}_+\to V\times Y,$$

we make the following additional assumption.

(A4) For each $1\le i\le n$, there exists a continuous map $\mathcal{P}_i:X\mapsto Z_i$, where (Z_i,Z_i^+) is an order Banach space with $Int(Z_i^+)\ne\emptyset$, such that

$$\mathcal{P}_iu(x_1,y,t)\gg\mathcal{P}_iu(x_2,y,t),\ \forall t>0,\ y\in Y,\ \text{whenever}\ x_1\ge x_2\ \text{with}\ \mathcal{P}_ix_1>\mathcal{P}_ix_2.$$

Theorem 3.4. *Assume that (A1) − (A4) hold. Then for any $(x_0,y_0)\in V\times Y$, $K=\omega(x_0,y_0)$ is an 1-covering of Y, and $\lim_{t\to\infty}\|u(x_0,y_0,t)-u(x_0^*,y_0,t)\|=0$, where $(x_0^*,y_0)=\omega(x_0,y_0)\cap Q^{-1}(y_0)$.*

Proof. By Theorem 2.2, Theorem 2.3 (2), and the assumptions (A2)-(A3), we can deduce that $K:=\omega(x_0,y_0)$ has a flow extension which is minimal and fiber distal, and the section map for K, $y\in Y\mapsto K_y$, is continuous at every $y\in Y$. Thus, Lemma 2.4 implies that $Y_0:=Y_0(K)=Y$ in our current case. Invoking Theorem 2.5, we conclude that for each $\hat{y}\in Y$,

$$\omega(p(\hat{y}),\hat{y})=K^*:=\{(p_*(y),y):y\in Y\}, \tag{4}$$

where $(p(\hat{y}),\hat{y}) = \text{g.l.b. of } K \cap Q^{-1}(\hat{y})$ and $(p_*(y),y) = \omega(p(\hat{y}),\hat{y}) \cap Q^{-1}(y)$ for every $y \in Y$. Let $(x,y) \in K$ be given arbitrary. Since K is minimal, there is $\tau_n \to +\infty$ such that $\Pi_{\tau_n}(x,y) \to (x,y)$ as $n \to \infty$. Note that $p(y) \le x$, $(p_*(y),y) = \omega(p(y),y) \cap Q^{-1}(y)$. On the other hand, $(A2)$ implies that there exists a subsequence of τ_n, which we relabel as τ_n such that $\Pi_{\tau_n}(p(y),y) \to (p_*(y),y)$. Hence $p_*(y) \le x$. As in the proof of [2, Theorem 4.1], we now prove that there is a subset $J \subset \{1,\cdots,n\}$, denoted by $J = \{1,2,\cdots,m\}$ without loss of generality, such that

$$\mathcal{P}_i p_*(y) = \mathcal{P}_i x \quad \text{for each } (x,y) \in K \text{ and } i \notin J, \tag{5}$$

$$\mathcal{P}_i p_*(y) \ll \mathcal{P}_i x \quad \text{for each } (x,y) \in K \text{ and } i \in J, \tag{6}$$

and that there exists an $\varepsilon_0 > 0$ such that

$$\mathcal{P}_i(B_+^J(p_*(y),\varepsilon_0)) \ll \mathcal{P}_i x \quad \text{for each } (x,y) \in K \text{ and } i \in J, \tag{7}$$

where $B_+^J(p_*(y),\varepsilon_0)$ denotes the set

$$\{x = (x_1,\cdots,x_m,p_{*m+1},\cdots,p_{*n}) \in V : x \ge p_*(y) \text{ and } \|x - p_*(y)\| \le \varepsilon_0\}.$$

We first show that if $\mathcal{P}_i p_*(\tilde{y}) = \mathcal{P}_i \tilde{x}$ for some $i \in \{1,\cdots,n\}$ and $(\tilde{x},\tilde{y}) \in K$, then $\mathcal{P}_i p_*(y) = \mathcal{P}_i x$ for any $(x,y) \in K$. Thanks to Theorem 2.2, we know that K admits a flow extension and $\mathcal{P}_i u(p_*(\tilde{y}),\tilde{y},s) = \mathcal{P}_i u(\tilde{x},\tilde{y},s)$ for all $s \in (-\infty,0)$. Otherwise, there would be $s \in (-\infty,0)$ with $\mathcal{P}_i u(p_*(\tilde{y}),\tilde{y},s) < \mathcal{P}_i u(\tilde{x},\tilde{y},s)$. Then $(A4)$ implies that $\mathcal{P}_i p_*(\tilde{y}) \ll \mathcal{P}_i \tilde{x}$, a contradiction. Let $(x,y) \in K$ be given. Since K is minimal, there exists a sequence $s_n \downarrow -\infty$ such that $\sigma_{s_n}(\tilde{y}) \to y$ and $u(\tilde{x},\tilde{y},s_n) \to x$. In view of Theorem 2.5 (3), we have

$$\mathcal{P}_i x = \lim_{n\to\infty} \mathcal{P}_i u(\tilde{x},\tilde{y},s_n) = \lim_{n\to\infty} \mathcal{P}_i u(p_*(\tilde{y}),\tilde{y},s_n) = \lim_{n\to\infty} \mathcal{P}_i p_*(\sigma_{s_n}(\tilde{y})) = \mathcal{P}_i p_*(y).$$

Thus, we have $\mathcal{P}_i x = \mathcal{P}_i p_*(y)$ for each $(x,y) \in K$.

Now we prove that if $\mathcal{P}_j p_*(\tilde{y}) < \mathcal{P}_j \tilde{x}$ for some $j \in \{1,\cdots,n\}$, then we have $\mathcal{P}_j p_*(y) \ll \mathcal{P}_j x$ for any $(x,y) \in K$. The flow extension on K implies $\mathcal{P}_j u(p_*(\tilde{y}),\tilde{y},s) < \mathcal{P}_j u(\tilde{x},\tilde{y},s)$ for all $s \in (-\infty,0)$. Suppose not, then there would be $s \in (-\infty,0)$ with $\mathcal{P}_j u(p_*(\tilde{y}),\tilde{y},s) = \mathcal{P}_j u(\tilde{x},\tilde{y},s)$. By Theorem 2.5 (3) and the above argument, we see that $\mathcal{P}_j p_*(\tilde{y}) = \mathcal{P}_j \tilde{x}$, a contradiction. Note that for any $(x,y) \in K$, there exists a sequence $s_n \downarrow -\infty$ such that $\sigma_{s_n}(\tilde{y}) \to y$ and $u(\tilde{x},\tilde{y},s_n) \to x$. Let $t_0 > 0$ be given, it is clear that $\sigma_{s_n - t_0}(\tilde{y}) \to \sigma_{-t_0}(y)$, $u(\tilde{x},\tilde{y},s_n - t_0) \to \bar{x} \in K_{\sigma_{-t_0}(y)}$ and $u(\bar{x},\sigma_{-t_0}(y),t_0) = x$. By Theorem 2.5 (3) again, we have

$$\begin{aligned}\mathcal{P}_j \bar{x} &= \lim_{n\to\infty} \mathcal{P}_j u(\tilde{x},\tilde{y},s_n - t_0) \ge \lim_{n\to\infty} \mathcal{P}_j u(p_*(\tilde{y}),\tilde{y},s_n - t_0)\\ &= \lim_{n\to\infty} \mathcal{P}_j p_*(\sigma_{s_n - t_0}(\tilde{y})) = \mathcal{P}_j p_*(\sigma_{-t_0}(y)).\end{aligned}$$

By the above result, we can deduce that $\mathcal{P}_j \bar{x} > \mathcal{P}_j p_*(\sigma_{-t_0}(y))$. Hence, (A4) and Theorem 2.5 (3) imply that

$$\mathcal{P}_j x = \mathcal{P}_j u(\bar{x},\sigma_{-t_0}(y),t_0) \gg \mathcal{P}_j u(p_*(\sigma_{-t_0}(y)),\sigma_{-t_0}(y),t_0) = \mathcal{P}_j p_*(y).$$

Since (6) implies (7), (5)–(7) follow immediately from the above arguments.

Let $(x,y) \in K$ and define $x_\alpha = (1-\alpha)p_*(y) + \alpha x$ for $\alpha \in [0,1]$, and

$$L = \{\alpha \in [0,1] : \omega(x_\alpha,y) = K^*\}.$$

Invoking Theorem 2.5, we see that $\omega(p_*(y),y) = K^*$. Combining this with the monotonicity of the semiflow, we deduce that if $0 < \alpha \in L$, then $[0,\alpha] \subset L$.

Next we show that L is closed, that is, if $[0,\alpha) \subset L$, then $\alpha \in L$. Since $\{\Pi(x_\alpha,y,t) : t \ge 0\}$ is uniformly stable, let $\delta(\varepsilon) > 0$ be the modulus of uniform stability for ε. Thus, we take $\beta \in [0,\alpha)$ with $\|x_\alpha - x_\beta\| < \delta(\varepsilon)$ and we obtain $\|u(x_\alpha,y,t) - u(x_\beta,y,t)\| < \varepsilon$ for each $t \ge 0$. Moreover, $\omega(x_\beta,y) = K^*$ and hence, there is a t_0 such that $\|u(x_\beta,y,t) - p_*(\sigma_t(y))\| < \varepsilon$ for each $t \ge t_0$. Then, we deduce that $\|u(x_\alpha,y,t) - p_*(\sigma_t(y))\| < 2\varepsilon$ for each $t \ge t_0$ and $\omega(x_\alpha,y) = K^*$, as claimed.

Now we prove that $L = [0,1]$. Assume, by contradiction, that $L = [0,\alpha]$ for some $0 \le \alpha < 1$. Let ε_0 be the number defined in $B_+^J(p_*(y),\varepsilon_0)$ of (7). Then the uniform stability assumption implies that we can take $\alpha < \gamma < 1$ such that

$$\|u(x_\alpha,y,t) - u(x_\gamma,y,t)\| < \frac{\varepsilon_0}{2}, \quad \forall t \ge 0. \tag{8}$$

As above, from $\omega(x_\alpha,y) = K^*$ we deduce that there is a $t_1 \ge 0$ such that $\|u(x_\alpha,y,t) - p_*(\sigma_t(y))\| < \frac{\varepsilon_0}{2}$ for each $t \ge t_1$. Consequently, for each $t \ge t_1$,

$$\|u(x_\gamma,y,t) - p_*(\sigma_t(y))\| < \varepsilon_0. \tag{9}$$

Let $(\tilde{x},\tilde{y}) \in \omega(x_\gamma,y)$, i.e., $(\tilde{x},\tilde{y}) = \lim_{n\to\infty}(u(x_\gamma,y,t_n),\sigma_{t_n}(y))$ for some $t_n \uparrow \infty$. The monotonicity, $p_*(y) \le x_\gamma$ and Theorem 2.5 (3) imply that $p_*(\sigma_{t_n}(y)) \le u(x_\gamma,y,t_n)$, which yields to $p_*(\tilde{y}) \le \tilde{x}$. Since $p_*(y) \le x_\gamma \le x$, we have $p_*(\sigma_{t_n}(y)) \le u(x_\gamma,y,t_n) \le u(x,y,t_n)$, and hence, by (5), we deduce that $\mathcal{P}_i p_*(\sigma_{t_n}(y)) = \mathcal{P}_i u(x_\gamma,y,t_n)$ for each $i \notin J$. This yields to $\mathcal{P}_i p_*(\tilde{y}) = \mathcal{P}_i \tilde{x}$ for each $i \notin J$. For any given $(z,\tilde{y}) \in K$, it follows from (5) that $\mathcal{P}_i p_*(\tilde{y}) = \mathcal{P}_i z$ for each $i \notin J$. By (9), we deduce that $\tilde{x} \in B_+^J(p_*(\tilde{y}),\varepsilon_0)$. In view of (7), we have $\mathcal{P}_i \tilde{x} \ll \mathcal{P}_i z$ for each $i \in J$. Thus, we can conclude that $p_*(\tilde{y}) \le \tilde{x} \le z$. Since this holds for each $(z,\tilde{y}) \in K$, the definition of p provides $p_*(\tilde{y}) \le \tilde{x} \le p(\tilde{y})$. From (4) we see that $\omega(p(\tilde{y}),\tilde{y}) = K^*$. It then follows from Theorem 2.5 that $\omega(\tilde{x},\tilde{y}) = K^* \subseteq \omega(x_\gamma,y)$. By (A3) and Theorem 2.3, we conclude that $\omega(\tilde{x},\tilde{y}) = \omega(x_\gamma,y) = K^*$, and hence $\gamma \in L$, a contradiction.

Since $L = [0,1]$, we have $\omega(x,y) = K^*$, and hence, the minimality of K implies that $K = K^*$ and $J = \emptyset$. Thus, K is an 1-covering of Y. As in the proof of Theorem 3.3, we can deduce that $\lim_{t\to\infty} \|u(x_0,y_0,t) - u(x_0^*,y_0,t)\| = 0$, where $(x_0^*,y_0) = \omega(x_0,y_0) \cap Q^{-1}(y_0)$. □

4 Applications

In this section, we applied the results in Sect. 3 to study the asymptotic recurrence of solutions to two recurrent evolution systems.

First, we consider

$$\frac{du}{dt} = A(t)u + f(t), \quad u(0) \in \mathbb{R}^n, \tag{10}$$

where $A(t)$ is a continuous $n \times n$ matrix function, and $f = (f_1, \ldots, f_n) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function. We assume that there exists $\delta_0 > 0$ such that

$$A_{ij}(t) \geq \delta_0, \quad f_i(t) \geq \delta_0, \ \forall 1 \leq i \neq j \leq n.$$

Define $A_s(t) = A(s+t)$ and $f_s(t) = f(s+t)$. We further assume that the set $\{(A_s, f_s) : s \in \mathbb{R}\}$ has compact closure $H(A, f)$ with respect to the compact open topology, and that the flow $\sigma : H(A,f) \times \mathbb{R} \to H(A,f)$, defined by $\sigma((B,g),t) = (B_t, g_t), t \in \mathbb{R}$, is minimal. For any given $(B,g) \in H(A,f)$, let $u(t,x,B,g), t \geq 0$, be the unique solution of the linear system $\frac{du}{dt} = A(t)u + f(t)$ satisfying $u(0) = x \in \mathbb{R}^n$.

Theorem 4.1. *Assume that for any $(B,g) \in H(A,f)$ and $x \in \mathbb{R}^n_+$, the solution $u(t,x,B,g)$ is bounded. Then (10) has a unique positive, recurrent and bounded full solution $u^*(t)$ such that $\lim_{t\to\infty} |u(t,x,A,f) - u^*(t)| = 0$ for any $x \in \mathbb{R}^n_+$.*

Proof. For each $x \in \mathbb{R}^n_+$ and $(B,g) \in H(A,f)$, let $u(t,x,B,g)$ be the unique solution of (10) with (A,f) replaced by (B,g). By the comparison theorem for cooperative systems, each $u(t,x,B,g)$ exists globally on $[0,\infty)$ and $u(t,x,B,g) \geq 0, \forall t \geq 0$. We define the skew-product semiflow Π_t on $\mathbb{R}^n_+ \times H(A,f)$ by $\Pi_t(x,(B,g)) = (u(t,x,B,g), \sigma_t(B,g))$. By the comparison theorem for cooperative and irreducible systems and the variation of constants formula for inhomogeneous linear systems, it then follows that the skew-product semiflow Π_t is strongly monotone and strongly subhomogeneous. By our assumption on f, we see that the omega limit set $\omega(x,A,f)$ is compact and $\omega(x,A,f) \subset Int(\mathbb{R}^n_+) \times H(A,f)$ for any $x \in \mathbb{R}^n_+$. Thus, Theorem 3.3 implies that for any $x \in \mathbb{R}^n_+$, $\omega(x,A,f)$ is a 1-covering of $H(A,f)$. Clearly, $\Pi_t : \omega(x,A,f) \to \omega(x,A,f)$ is a compact, minimal and fiber distal flow.

Let $x^0 \in \mathbb{R}^n_+$ be given, and define $u^*(t) := u(t,x^*,A,f)$, where $(x^*,(A,f)) = \omega(x^0,A,f) \cap Q^{-1}(A,f)$. It then follows that $u^*(t)$ is a positive, recurrent and bounded full solution of (10). In order to prove that $\omega(x,A,f) = \omega(x^0,A,f)$ for any $x \in \mathbb{R}^n_+$, by the minimality of both $\Pi_t : \omega(x,A,f) \to \omega(x,A,f)$ and $\Pi_t : \omega(x^0,A,f) \to \omega(x^0,A,f)$, it suffices to prove that $\omega(x,A,f) \cap \omega(x^0,A,f) \neq \emptyset$. Assume, by contradiction, that $\omega(x,A,f) \cap \omega(x^0,A,f) = \emptyset$. Then we have $d(\omega(x,A,f), \omega(x^0,A,f)) > 0$, where d is the metric on the product space $\mathbb{R}^n_+ \times H(A,f)$. Let $(x_1,B,g) = \omega(x,A,f) \cap Q^{-1}(B,g)$ and $(x_2,B,g) = \omega(x^0,A,f) \cap Q^{-1}(B,g)$. On the other hand, by the minimality and the 1-covering property, there exists $t_n \to \infty$ such that $\lim_{n\to\infty} \Pi_{t_n}(x_i,(B,g)) = (x_i,(B,g))$, and hence $\lim_{n\to\infty} u(t_n,x_i,B,g) = x_i$, $i = 1,2$. For any fixed $t_0 > 0$, let

$$(x_1', (B,g)') = \Pi_{t_0}(x_1,(B,g)) \in \omega(x,A,f) \subset Int(\mathbb{R}^n_+) \times H(A,f),$$

and

$$(x_2', (B,g)') = \Pi_{t_0}(x_2,(B,g)) \in \omega(x^0,A,f) \subset Int(\mathbb{R}^n_+) \times H(A,f).$$

Then

$$\Pi_{t_n}(x_i,(B,g)) = \Pi_{t_n - t_0}(x_i', (B,g)'), \quad \forall i = 1,2, n \geq 1.$$

By the strong monotonicity and strong subhomogeneity of Π_t, Claim 2 in the proof [14, Theorem 2.1], and [14, Remarks 2.1–2.2], we then obtain

$$\begin{aligned}
\rho(x_1,x_2) &= \lim_{n\to\infty} \rho(u(t_n,x_1,B,g), u(t_n,x_2,B,g)) \\
&= \lim_{n\to\infty} \rho(u(t_n - t_0, x_1', (B,g)'), u(t_n - t_0, x_2', (B,g)')) \\
&\leq \rho(x_1', x_2') = \rho(u(t_0,x_1,B,g), u(t_0,x_2,B,g)) \\
&< \rho(x_1,x_2),
\end{aligned}$$

a contradiction. Therefore, $\omega(x,A,f) = \omega(x^0,A,f)$ for any $x \in \mathbb{R}^n_+$. It then follows from Theorem 3.3 that $\lim_{t\to\infty} |u(t,x,A,f) - u^*(t)| = 0$ for any $x \in \mathbb{R}^n_+$. □

Next, we consider the scalar nonautonomous Kolmogorov parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = d(t)\triangle u + uf(x,t,u) & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{11}$$

where Ω is a bounded and open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Δ is the Laplacian operator on $\mathbb{R}^N$, $Bv = v$ or $Bv = \frac{\partial v}{\partial n} + \alpha v$ for some nonnegative function $\alpha \in C^{1+\theta}(\partial\Omega, \mathbb{R})$, $d(\cdot) \in C(\mathbb{R},\mathbb{R})$, and $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R})$.

Let $H(d,f)$ be the closure of $\{(d_s,f_s) : s \in \mathbb{R}\}$ with respect to the compact open topology, where $(d_s,f_s) \in C(\mathbb{R},\mathbb{R}) \times C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ is defined by

$$d_s(t) = d(s+t), \quad f_s(x,t,u) = f(x,t+s,u), \ \forall (x,t,u) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+.$$

Define $\sigma_t(\mu,g) = (\mu_t, g_t)$, $(\mu,g) \in H(d,f)$, $t \in \mathbb{R}$. We assume that

(B1) $H(d,f)$ is compact with respect to the compact open topology and the flow σ_t on $H(d,f)$ is minimal.

(B2) $d(\cdot) \in C(\mathbb{R},\mathbb{R})$ is bounded with $d(t) \geq d_0$, $\forall t \in \mathbb{R}$, for some $d_0 > 0$, and $d(t)$ is Hölder continuous in $t \in \mathbb{R}$; $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ is bounded, $f_u'(x,t,u) \leq 0$, $\forall (x,t,u) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}_+$, and $f(x,t,0)$ is uniformly Hölder continuous in $(t,x) \in \mathbb{R} \times \bar{\Omega}$.

(B3) There exists $M_0 > 0$ such that $f(x,t,M_0) \leq 0$, $\forall (x,t) \in \bar{\Omega} \times \mathbb{R}$.

Let $p \in (N,\infty)$ be fixed. For each $\beta \in (1/2+N/(2p),1)$, let X_β be the fractional power space of $X = L^p(\Omega)$ with respect to $(-\Delta, B)$. Then X_β is an ordered Banach space with the cone X_β^+ consisting of all nonnegative functions in X_β, and X_β^+ has nonempty interior $Int(X_\beta^+)$. Moreover, $X_\beta \subset C^{1+\nu}(\bar{\Omega})$ with continuous inclusion for $\nu \in [0, 2\beta - 1 - N/p)$. We denote the norms in X_β and $L^2(\Omega)$ by $\|\cdot\|_\beta$ and $\|\cdot\|_2$, respectively.

By the theory of semilinear parabolic differential equations (see, e.g., [1, Sect. III. 20]), it follows that for every $\phi \in X_\beta^+$ and $(\mu,g) \in H(d,f)$, the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu(t)\triangle u + ug(x,t,u) & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = \phi \end{cases} \tag{12}$$

has a unique regular solution $u(x,t,\phi,\mu,g)$ with the maximal interval of existence $I(\phi,\mu,g) \subset [0,\infty)$, and $I(\phi,\mu,g) = [0,\infty)$ provided $u(\cdot,t,\phi,\mu,g)$ has an L^∞-bound on $I(\phi,\mu,g)$.

According to [4,5], the principal spectrum of the linear nonautonomous parabolic problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = d(t)\Delta v + f(x,t,0)v, & x \in \Omega,\ t \in \mathbb{R}, \\ Bv = 0, & x \in \partial\Omega,\ t \in \mathbb{R} \end{cases} \tag{13}$$

is defined to be the dynamical (Sacker-Sell) spectrum of its associated linear skew-product flow restricted to the one-dimensional subbundle of $X_\beta \times H(d,f(\cdot,\cdot,0))$. By [4, Theorem 2.6 and Proposition 2.11 (i)], it follows that the principal spectrum of (13) is a nonempty and compact interval $[\lambda_{\inf}(d,f(\cdot,\cdot,0)), \lambda_{\sup}(d,f(\cdot,\cdot,0))]$, and (13) admits a unique strongly positive full solution $v(t)$, $t \in \mathbb{R}$, with $\|v(0)\|_2 = 1$ such that

$$\lambda_{\inf}(d,f(\cdot,\cdot,0)) = \liminf_{t-s\to\infty} \frac{\log\|v(t)\|_2 - \log\|v(s)\|_2}{t-s},$$

and

$$\lambda_{\sup}(d,f(\cdot,\cdot,0)) = \limsup_{t-s\to\infty} \frac{\log\|v(t)\|_2 - \log\|v(s)\|_2}{t-s}.$$

We are now in a position to prove the following result on the global convergence for (11).

Theorem 4.2. *Let (B1)–(B3) hold. Then the following two statements are valid:*

(1) If $\lambda_{\sup}(d,f(\cdot,\cdot,0)) < 0$, then $\lim_{t\to\infty}\|u(\cdot,t,\phi,d,f)\|_\beta = 0$ for every $\phi \in X_\beta^+$;

(2) If $\lambda_{\inf}(d,f(\cdot,\cdot,0)) > 0$, then for every $\phi \in X_\beta^+ \setminus \{0\}$, there exists a positive, recurrent and bounded full solution $u(x,t,\phi^*,d,f)$ of (11) such that $\lim_{t\to\infty}\|u(\cdot,t,\phi,d,f) - u(\cdot,t,\phi^*,d,f)\|_\beta = 0$.

Proof. For any $(\mu,g) \in H(d,f)$, both $(B2)$ and $(B3)$ imply that $u = M$, $M \geq M_0$, is an upper-solution of (12), and hence, by the comparison theorem and a priori estimates of parabolic equations (see, e.g., [1]), each solution $u(x,t,\phi,\mu,g)$ exists globally on $[0,\infty)$, and for any $t_0 > 0$, the set $\{u(\cdot,t,\phi,\mu,g) : t \geq t_0\}$ is precompact in X_β^+. We define the skew-product semiflow $\Pi_t : X_\beta^+ \times H(d,f) \to X_\beta^+ \times H(d,f)$ by $\Pi_t(\phi,\mu,g) = (u(\cdot,t,\phi,\mu,g),\mu_t,g_t)$. Then for each $(\phi,\mu,g) \in X_\beta^+ \times H(d,f)$, the omega limit set $\omega(\phi,\mu,g)$ of the forward orbit $\gamma^+(\phi,\mu,g) := \{\Pi_t(\phi,\mu,g) : t \geq 0\}$ is well defined, compact and invariant under Π_t, $t \geq 0$. Moreover, the maximum principle for parabolic equations implies that $\Pi_t((X_\beta^+ \setminus \{0\}) \times H(d,f)) \subset Int(X_\beta^+) \times H(d,f)$, $\forall t > 0$.

In the case where $\lambda_{\sup}(d,f(\cdot,\cdot,0)) < 0$, we choose a sufficiently small number $\delta_1 > 0$ such that $\lambda_{\sup}(d,f(\cdot,\cdot,0)) + \delta_1 < 0$. It then follows that $\|v(t)\|_2 \leq e^{(\lambda_{\sup}(d,f(\cdot,\cdot,0))+\delta_1)t}$ for sufficiently large t, and hence $\lim_{t\to\infty}\|v(t)\|_2 = 0$. For any $\phi \in X_\beta^+$, there exists a sufficiently large number $K > 0$ such that $\phi \leq Kv(0)$. Note that $u(x,t,\phi,d,f)$ satisfies the following differential inequality

$$\begin{cases} \dfrac{\partial u}{\partial t} \leq d(t)\triangle u + f(x,t,0)u & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{14}$$

By the comparison principle, we see that

$$u(\cdot,t,\phi,d,f) \leq Kv(t),\ \forall t \geq 0,$$

and hence, $\lim_{t\to\infty}\|u(\cdot,t,\phi,d,f)\|_2 = 0$. Now we prove $\lim_{t\to\infty}\|u(\cdot,t,\phi,d,f)\|_\beta = 0$. For any $(\psi,\mu,g) \in \omega(\phi,d,f)$, there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty}\Pi_{t_n}(\phi,d,f) = (\psi,\mu,g)$, and hence, $\lim_{n\to\infty}\|u(\cdot,t_n,\phi,d,f) - \psi\|_\beta = 0$. Since $X_\beta \subset C^1(\bar{\Omega})$ with continuous inclusion, we have $\lim_{n\to\infty} u(x,t_n,\phi,d,f) = \psi(x)$ uniformly for $x \in \bar{\Omega}$, and hence, $\|\psi\|_2 = 0$. Since $\psi(x)$ is nonnegative and continuous on $\bar{\Omega}$, we further obtain that $\psi(\cdot) \equiv 0$. It then follows that $\lim_{t\to\infty}\|u(\cdot,t,\phi,d,f)\|_\beta = 0$.

In the case where $\lambda_{\inf}(d,f(\cdot,\cdot,0)) > 0$, we fix a positive number $\varepsilon < \lambda_{\inf}(d,f(\cdot,\cdot,0))$. Then we have the following claim.

Claim. There exists $\delta > 0$ such that $\limsup_{t\to\infty}\|u(\cdot,t,\phi,\mu,g)\|_\beta \geq \delta$ for all $(\phi,\mu,g) \in (X_\beta^+ \setminus \{0\}) \times H(d,f)$.

Indeed, since $H(d,f)$ is compact and the translation flow σ_t is minimal on $H(d,f)$, there exists $\delta_0 > 0$ such that

$$|g(x,t,u) - g(x,t,0)| < \varepsilon, \quad \forall x \in \bar{\Omega},\ t \in \mathbb{R},\ u \in [0,\delta_0],\ (\mu,g) \in H(d,f).$$

Since $X_\beta \subset C^1(\bar{\Omega})$ with continuous inclusion, there exists $\delta > 0$ such that for any $\phi \in X_\beta$, $\|\phi\|_\beta \le \delta$ implies that $\|\phi\|_\infty \le \delta_0$. Suppose for contradiction that for some $(\phi,\mu,g) \in (X_\beta^+ \setminus \{0\}) \times H(d,f)$, there holds $\limsup_{t\to\infty} \|u(\cdot,t,\phi,\mu,g)\|_\beta < \delta$. Then there is $t_0 > 0$ such that $\|u(\cdot,t,\phi,\mu,g)\|_\beta < \delta$ for all $t \ge t_0$, and hence $\|u(\cdot,t,\psi,\gamma,h)\|_\beta < \delta$ for all $t \ge 0$, where $(\psi,\gamma,h) = (u(\cdot,t_0,\phi,\mu,g),\mu_{t_0},g_{t_0}) \in Int(X_\beta^+) \times H(d,f)$. By the choice of δ_0 and δ, it follows that $u(x,t,\psi,\gamma,h)$ satisfies the following differential inequality

$$\begin{cases} \dfrac{\partial u}{\partial t} \ge \gamma(t)\triangle u + (h(x,t,0) - \varepsilon)u & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{15}$$

It is easy to check that $(\gamma,h(\cdot,\cdot,0)) \in H(d,f(\cdot,\cdot,0))$ and the translation flow σ_t is minimal on $H(d,f(\cdot,\cdot,0))$. Thus, we have $H(\gamma,h(\cdot,\cdot,0)) = H(d,f(\cdot,\cdot,0))$. It then follows that

$$\lambda_{\inf}(\gamma,h(\cdot,\cdot,0)) = \lambda_{\inf}(d,f(\cdot,\cdot,0)) > \varepsilon.$$

Let $w(t)$ be the unique strongly positive full solution of the linear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma(t)\triangle u + h(x,t,0)u & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{16}$$

We choose a sufficiently small number $\delta_2 > 0$ such that $\lambda_{\inf}(\gamma,h(\cdot,\cdot,0)) > \varepsilon + \delta_2$. By [4, Proposition 2.11 (i)], it then follows that $\|w(t)\|_2 \ge e^{(\lambda_{\inf}(\gamma,h(\cdot,\cdot,0))-\delta_2)t}$ for sufficiently large t, and hence, $\lim_{t\to\infty} \|e^{-\varepsilon t}w(t)\|_2 = \infty$. Since $\psi \gg 0$, there exists a sufficiently small number $k > 0$ such that $\psi \ge kw(0)$. By (15) and the comparison principle, it follows that

$$u(\cdot,t,\psi,\gamma,h) \ge ke^{-\varepsilon t}w(t), \ \forall t \ge 0,$$

and hence, $\lim_{t\to\infty} \|u(\cdot,t,\psi,\gamma,h)\|_2 = \infty$, which contradicts the boundedness of $u(\cdot,t,\psi,\gamma,h)$ in $C(\bar{\Omega})$.

By the claim above and the same arguments as in [14, Theorem 3.1], we further have

$$\omega(\phi,\mu,g) \subset Int(X_\beta^+) \times H(d,f), \quad \forall (\phi,\mu,g) \in (X_\beta^+ \setminus \{0\}) \times H(d,f).$$

Let $u(\phi,\mu,g,t) := u(\cdot,t,\phi,\mu,g), t \ge 0$. By the standard comparison theorem, it then follows that $u(\cdot,\mu,g,t)$ is strongly monotone on X_β^+ for each $(\mu,g,t) \in H(d,f) \times (0,\infty)$. It is easy to see from $(B2)$ that each function $ug(x,t,u)$, $(\mu,g) \in H(d,f)$, is subhomogeneous in u for any fixed $(x,t) \in \bar{\Omega} \times \mathbb{R}_+$. By the integral version of parabolic (12) (see, e.g., [1]), it then follows that $u(\cdot,\mu,g,t)$ is subhomogeneous on X_β^+ for each $(\mu,g,t) \in H(d,f) \times \mathbb{R}_+$. Thus, the skew-product semiflow Π_t is subhomogeneous and strongly monotone on $X_\beta^+ \times H(d,f)$. By Theorem 3.3, it follows that for every $\phi \in X_\beta^+ \setminus \{0\}$, $\omega(\phi,d,f)$ is a 1-covering of $H(d,f)$, and $\lim_{t\to\infty} \|u(\cdot,t,\phi,d,f) - u(\cdot,t,\phi^*,d,f)\|_\beta = 0$, where $(\phi^*,d,f) \in \omega(\phi,d,f)$. Since $\Pi_t : \omega(\phi,d,f) \to \omega(\phi,d,f)$ is a minimal flow, $u(\cdot,t,\phi^*,d,f)$ is a positive, recurrent and bounded solution of (11). □

Finally, we remark that Theorem 3.4 can be applied to establish the global asymptotic recurrence of bounded solutions for n-dimensional monotone and recurrent nonautonomous differential systems with a first integral, which generalizes the results on the almost periodicity for these systems obtained in [2, 11]. For such a system with infinite time delay, one may choose the phase space X to be an appropriate subset of $C((-\infty,0],\mathbb{R}^n)$ and the ordered Banach space (Z_i,Z_i^+) to be $(\mathbb{R},\mathbb{R}^+)$, and define $\mathcal{P}_i(\phi) = \phi_i(0)$ in order to verify the assumption (A4) for the associated skew-product semiflow.

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