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Asymptotic Behavior for Asymptotically Periodic Semiflows with Applications[1]

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Abstract

The properties of ω-limit set and global asymptotic behavior are first obtained for asymptotically autonomous discrete dynamical processes on metric spaces.Then certain equivalence of the asymptotic behavior between an asymptotically periodic semiflows and its associated asymptotically autonomous discrete dynamical process is proved. As some applications, the global behavior of asymptotically periodic scalar Kolmogorov parabolic equations and predator-prey parabolic systems are also discussed.

Key words and phrases : asymptotically periodic semiflow, discrete dynamical process, ω-limit set, global asymptotic behavior and nonautonomous parabolic equations.

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§1. Introduction

Recently, there have been a series of important investigations on asymptotically autonomous differential equations and asymptotically autonomous semiflows on metric spaces([3,16,19-21]). The motivation of this study partially comes from the convergence problem in certain chemostat/gradostat and epidemic models. In particular, Thieme's examples [19,20] show that the solutions of asymptotically autonomous differential equations don't have the same asymptotic behavior as the solutions of the associated limit equation in general. A natural consideration of a periodically varying environment(e.g.,

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the seasonal fluctuations and periodic washout rate in a chemostat) leads to the models of periodic systems of differential equations. However, in some situations, the varying parameters in the modelling systems could be non-periodic with respect to time t, but tend to periodic ones with the evolution of time, which leads to the so-called asymptotically periodic nonautonomous systems of differential equations. Moreover, due to a certain conservation principle(see, e.g., [17]), a periodic chemostat/gradostat system could be reduced to asymptotically periodic systems. In [13], Hess studied the global asymptotic behavior of the asymptotically periodic Fisher reaction-diffusion equation, a well-known model of population genetics. By a certain Gronwall's inequality argument it easily follows that solutions of an asymptotically periodic system of differential equations generate an asymptotically periodic semiflow(see Definition 3.1 and Propositions 3.1 and 3.2). Then, in this paper, we study the asymptotic behavior of asymptotically periodic semiflows on metric spaces with the aim at wide-ranging applications. In our investigation, it is observed that an asymptotically periodic semiflow induces naturally an asymptotically autonomous discrete dynamical process with the limit autonomous discrete semiflow(see Definition 2.1). This further motivates us to focus on the asymptotically autonomous discrete dynamical processes. A somewhat related study is Fujimoto and Krause's recent research on the inhomogeneous iterations of nonlinear operators on both Euclidean and Banach spaces ([6,7]). Under some appropriate assumptions, it easily follows that the inhomogeneous iteration of the rescaled nonlinear operators is actually an asymptotically autonomous discrete dynamical process in the sense of Definition 2.1.

This paper is organized as follows. In Section 2, the definition of asymptotically autonomous discrete dynamical processes is given and the properties of ω-limit set (Theorem 2.1) and a Butler-McGehee type lemma(Theorem 2.2) for asymptotically autonomous discrete dynamical processes are proved by embedding an asymptotically autonomous discrete dynamical process and its limit discrete semiflow into an autonomous discrete semiflow. Then, based on Theorem 2.2, we further obtain the global results on the ω-limit set(Theorem 2.3), convergence (Theorem 2.4)and uniform persistence(and repellor)(Theorem 2.5). In Section 3, the definition of asymptotically periodic semiflows is given. Then we prove the main result of this section(Theorem 3.1), which confirms the reduction of the study of asymptotic behavior of an asymptotically periodic semiflow into that of its associated asymptotically autonomous discrete dynamical process. The sufficient conditions for nonautonomous parabolic systems and systems of ordinary differential equations to generate asymptotically periodic semiflows are also provided (Propositions 3.1 and 3.2). In Section 4, as some applications of general results in Sections 2 and 3, the threshold result on the global behavior of the solutions of asymptotically periodic scalar Kolmogorov parabolic equations(Theorem 4.1) and the uniform persistence for the asymptotically periodic predator-prey parabolic systems(Theorem 4.2) are proved.

§2. Asymptotically Autonomous Discrete Dynamical Processes

Let (X, d) be a metric space. By a sequence of continuous mapping $S_m : X \to X,\ m \in N$, we define a discrete dynamical process by

$$\begin{cases} T_n = S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0 : X \to X, & n \geq 1 \\ T_0 = I \end{cases}$$

where $I : X \to X$ is the identical mapping on X. For $x \in X$, we let $\gamma^+(x) = \{T_n(x); n \geq 0\}$ denote its orbit, and $\omega(x) = \{y; y \in X$ and there is $n_k \to \infty$ such that $T_{n_k}(x) \to y$ as $k \to \infty\}$ its ω-limit set.

Definition 2.1 Let $T_n : X \to X, n \geq 0$ be a discrete dynamical process and $S : X \to X$ a continuous map. T_n $(n \geq 0)$ is called asymptotically autonomous-with limit discrete semiflow S^n $(n \geq 0)$ - if

$$S_{m_j}(x_j) \to S(x),\ j \to \infty,$$

for any two sequences $m_j \to \infty, x_j \to x$ $(j \to \infty)$ with $x, x_j \in X$.

Throughout this section, we always assume that $T_n : X \to X,\ n \geq 0$ is an asymptotically autonomous discrete dynamical process with limit discrete semiflow $S^n : X \to X, n \geq 0$.

Let $\overline{N} = N \cup \{\infty\}$. For any given strictly increasing continuous function $\phi : [0, \infty) \to [0, 1)$ with $\phi(0) = 0$ and $\phi(\infty) = 1 (e.g., \phi(s) = \frac{s}{1+s})$, we can define a metric ρ on $\overline{N}$ as $\rho(m_1, m_2) = |\phi(m_1) - \phi(m_2)|$, for any $m_1, m_2 \in \overline{N}$, and then $\overline{N}$ is compactified. Motivated by Thieme's observation ([19]) for asymptotically autonomous semiflows, in order to embed $T_n : X \to X, n \geq 0$ and $S^n : X \to X, n \geq 0$ into an autonomous discrete semiflow on the larger metric space $\overline{N} \times X = \widetilde{X}$, we define a mapping $\widetilde{S} : \widetilde{X} \to \widetilde{X}$ by

$$\widetilde{S}((m, x)) = \begin{cases} (1 + m, S_m(x)) & m < \infty,\ x \in X \\ (\infty, S(x)) & m = \infty,\ x \in X. \end{cases} \tag{2.1}$$

Then, by Definition 2.1, it easily follows that $\widetilde{S} : \widetilde{X} \to \widetilde{X}$ is continuous, and hence we have a discrete semiflow $\widetilde{S}^n$ $(n \geq 0)$ on $\widetilde{X}$ with

$$\widetilde{S}^n((m,x)) = \begin{cases} (m+n, S_{m+(n-1)} \circ \cdots \circ S_{m+1} \circ S_m(x), & m < \infty \\ (\infty, S^n(x)) & m = \infty. \end{cases}$$

In particular, let $m = 0$,

$$\widetilde{S}^n((0,x)) = (n, S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0(x)) = (n, T_n(x)), n \geq 0. \tag{2.2}$$

Clearly, by the compactness of $\overline{N}$ and (2.2), for any precompact orbit $\gamma^+(x)$ of T_n $(n \geq 0)$, the orbit $\gamma^+((0,x))$ of $\widetilde{S}^n$ $(n \geq 0)$ is precompact and

$$\{\infty\} \times \omega(x) = \omega((0,x)) \tag{2.3}$$

where $\omega((0,x))$ is the ω-limit set of $(0,x)$ for $\widetilde{S}^n$ $(n \geq 0)$ in the usual way.

Let $M \subseteq X$ be a S-invariant set under a continuous map $S : X \to X$, i.e., $S(M) = M$. According to [5, Definition 2.6], M is said to be compactly invariantly connected if whenever $M \subseteq M_1 \cup M_2$, where M_1 and M_2 are disjoint, nonempty, compact and invariant sets, then either $M_1 \cap M = \emptyset$ or $M_2 \cap M = \emptyset$.

By applying [15,Theorem 1.5.2] and [5, Proposition 2.1] to the discrete semiflow $\widetilde{S}^n : \widetilde{X} \to \widetilde{X}, n \geq 0$, together with the compactness of $\overline{N}$ and (2.2) and (2.3), we can easily derive the following result on the properties of the ω-limit set of $T_n : X \to X, n \geq 0$.

Theorem 2.1 Let the orbit $\gamma^+(x)$ of $T_n : X \to X, n \geq 0$ be precompact in X. Then its ω-limit set $\omega(x)$ has the following properties:

(a) $\omega(x)$ is nonempty and compact;

(b) $\omega(x)$ is S-invariant, i.e., $S(\omega(x)) = \omega(x)$, and compactly S-invariantly connected;

(c) $\omega(x)$ attracts $\gamma^+(x)$, i.e., $\lim_{n\to\infty} d(T_n(x), \omega(x)) = 0$.

For a given subset M of X, we call M positive $T_n(n \geq 0)$-invariant if $S_m(M) \subseteq M$ for all $m \geq 0$. According to [5, Definition 2.8], a nonempty and closed S-invariant subset M of X is an isolated S-invariant set if it is the maximal (under the order of inclusion) S-invariant set in some neighbourhood of itself.

Now we are in a position to prove a very useful Butler-McGehee type lemma for asymptotically autonomous discrete dynamical process. For a discrete semiflow version of it, we refer to [5, Theorem 3.1].

Theorem 2.2 Let M be an isolated S-invariant set in X, and let $\gamma^+(x)$ be an orbit of $T_n, n \geq 0$ and $\omega(x)$ its ω-limit set. Assume that $\gamma^+(x)$ is precompact in X, and that $\omega(x) \cap M \neq \emptyset$ but $\omega(x) \not\subseteq M$. Then

(a) there exists a $u \in \omega(x) \setminus M$ with its S-orbit $\gamma_S^+(u) \subseteq \omega(x)$ and ω-S-limit set $\omega_S(u) \subseteq M$, and

(b) there exists a $w \in \omega(x) \setminus M$ with a full S-orbit $\gamma_S(w) \subseteq \omega(x)$ and its α-S-limt set $\alpha_S(w) \subseteq M$.

Proof. Let $\widetilde{S} : \widetilde{X} \to \widetilde{X}$ be defined as in (2.1), then the $\widetilde{S}$-orbit $\gamma^+((0,x))$ is precompact, and its ω-$\widetilde{S}$-limit set $\tilde{\omega} = \omega((0,x)) = \{\infty\} \times \omega(x)$ (see (2.3)). Let $\widetilde{M} = \{\infty\} \times M$, then it easily follows that $\widetilde{M}$ is an isolated $\widetilde{S}$-invariant set of $\widetilde{X}$, and $\tilde{\omega} \cap \widetilde{M} = \{\infty\} \times (\omega(x) \cap M) \neq \emptyset$, but $\tilde{\omega} \not\subseteq \widetilde{M}$. By the Butler-McGehee lemma for discrete semiflows([5, Theorem 3.1]), it follows that

(i) there exists $\tilde{u} \in \omega \setminus \widetilde{M}$, $\gamma^+(\tilde{u}) \subseteq \tilde{\omega}$, and $\omega(\tilde{u}) \subseteq \widetilde{M}$, i.e., $\tilde{u} = (\infty, u)$ for some $u \in \omega(x) \setminus M$, $\gamma^+(\tilde{u}) = \{\infty\} \times \gamma_S^+(u) \subseteq \tilde{\omega} = \{\infty\} \times \omega(x)$, and $\omega(\tilde{u}) = \{\infty\} \times \omega_S(u) \subseteq \widetilde{M} = \{\infty\} \times M$, and hence $\gamma_S^+(u) \subseteq \omega(x)$ and $\omega_S(u) \subseteq M$;

(ii) there exists $\tilde{w} \in \tilde{\omega} \setminus \widetilde{M}$, and a full orbit $\{\tilde{w}_n, n \in Z\} \subseteq \tilde{\omega}$ with $\tilde{w}_0 = \tilde{w}$, $\tilde{w}_{n+1} = \widetilde{S}(\tilde{w}_n)$, for all $n \in Z$, and $\alpha(\tilde{w}) \subseteq \widetilde{M}$, i.e., $\tilde{w} = (\infty, w)$ for some $w \in \omega(x) \setminus M$, and $\tilde{w}_n = (\infty, w_n)$ for some $w_n \in \omega(x)$, $n \in Z$ with $(\infty, w_0) = (\infty, w)$, and $(\infty, w_{n+1}) = \widetilde{S}((\infty, w_n)) = (\infty, S(w_n))$ for all $n \in Z$, then $w_0 = w$ and $w_{n+1} = S(w_n)$ for all $n \in Z$, i.e., $\{w_n; n \in Z\}$ is a full S-orbit in $\omega(x)$. Since $\omega(x)$ is compact and S-invariant (by Theorem 2.1), the α-S-limit $\alpha_S(w)$ exists and $\alpha_S(w) \subseteq \omega(x)$. Since $\alpha(\tilde{w}) = \{\infty\} \times \alpha_S(w) \subseteq \widetilde{M} = \{\infty\} \times M$, $\alpha_S(w) \subseteq M$.

Clearly, (i) and (ii) imply (a) and (b), respectively, and this completes the proof.

Theorem 2.3 Let M be a compact S-invariant subset of X which is locally asymptotically stable for S and $W^s(M) = \{y \in X; \omega_S(y) \neq \emptyset$ and $\omega_S(y) \subseteq M\}$ be its stable set. Then for any precompact $T_n(n \geq 0)$-orbit $\gamma^+(x)$ with $\omega(x) \cap W^s(M) \neq \emptyset$, $\omega(x) \subseteq M$.

Proof. Recall that M is locally asymptotically stable for S if M is stable and M attracts points locally (see[8, Chapter 2.2]), then it easily follows that M is isolated. Let $y \in \omega(x) \cap W^s(M)$, then $y \in \omega(x)$ and $\omega_S(y) \subseteq M$. By the compactness and S-invariance of $\omega(x)$(Theorem 2.1), $\omega_S(y) \subseteq \omega(x)$ and hence $\omega_S(y) \subseteq \omega(x) \cap M$. Then $\omega(x) \cap M \neq \emptyset$. Assume that $\omega(x) \nsubseteq M$, then by Theorem 2.2, there exists $w \in \omega(x) \setminus M$ with a full S-orbit $\gamma_S(w) = \{w_n; n \in Z\} \subseteq \omega(x)$ and $\alpha_S(w) \subseteq M$. Since $w \notin M$, there exists a neighborhood V of M such that $w \notin V$. Then, by the stability of M(see[8, Chapter 2.2]), there exists a neighborhood U of M such that $S^n U \subseteq V$ for all $n \geq 0$. Since $\alpha_S(w) \subseteq M$, there exists a $n_0 > 0$ such that $w_{-n_0} \in U$, and hence, since $w_0 = w$ and $w_{n+1} = S(w_n)$ for all $n \in Z$, $w = w_0 = S^{n_0}(w_{-n_0}) \in V$, which contradicts $w \notin V$.

This completes the proof.

Then we prove the following two results on the convergence and uniform persistence(repellor) for the precompact orbits of $T_n, n \geq 0$. For some unexplained terminologies, we refer to [5,14,23].

Theorem 2.4 Assume that each fixed point of S is isolated, that there is no S-cyclic chain of fixed points of S, and that every precompact S-orbit converges to some fixed point of S. Then any precompact orbit $\gamma^+(x)$ of $T_n, n \geq 0$, converges to some fixed point of S.

Proof. Since $\omega(x)$ is nonempty, compact and S-invariant subset of X, there exists $y \in \omega(x)$ such that $\gamma_S^+(y) \subseteq \omega(x)$, and hence the convergence of $\gamma_S^+(y)$ implies that $\omega(x)$ contains some fixed point of S. Let $E = \{e; S(e) = e \text{ and } e \in \omega(x)\}$, the $E \neq \emptyset$ and, by the compactness of E and isolatedness of each fixed point of S, $E = \{e_1, e_2, \cdots, e_m\}$ for some integer $m > 0$. Assume that, by contradiction, $\omega(x)$ is no singleton. Since $E \neq \emptyset$, there exists some i_1 $(1 \leq i_1 \leq m)$ such that $e_{i_1} \in \omega(x)$, i.e., $\omega(x) \cap \{e_{i_1}\} \neq \emptyset$. Since $\omega(x) \nsubseteq \{e_{i_1}\}$, by Theorem 2.2, there exists $w_1 \in \omega(x) \setminus \{e_{i_1}\}$ and a full S-orbit $\gamma_S(w_1) \subseteq \omega(x)$ and $\alpha_S(w_1) = e_{i_1}$. Since $\gamma_S^+(w_1) \subseteq \omega(x)$, there exists some i_2 $(1 \leq i_2 \leq m)$ such that $\omega_S(w_1) = e_{i_2}$. Therefore, e_{i_1} is chained to e_{i_2}, i.e., $e_{i_1} \to e_{e_2}$. Since $\omega(x) \cap \{e_{i_2}\} \neq \emptyset$ and $\omega(x) \nsubseteq \{e_{i_2}\}$, again by Theorem 2.2 , there exists $w_2 \in \omega(x) \setminus \{e_{i_2}\}$ and a full S-orbit $\gamma_S(w_2) \subseteq \omega(x)$ and $\alpha_S(w_2) = e_{i_2}$. We can repeat the above argument to get an i_3 $(1 \leq i_3 \leq m)$ such that $e_{i_2} \to e_{i_3}$. Since there are only a finite number of e_i's, we will eventually arrive at a S-cyclic chain of some fixed points of S, which contradicts our assumption.

This completes the proof.

Theorem 2.5 Let X_0 and ∂X_0 be open and closed subsets of X, respectively, such that $X_0 \cap \partial X_0 = \emptyset$ and $X = X_0 \cup \partial X_0$. Assume that $S_m(X_0) \subseteq X_0$ for all $m \geq 0$, and $S(X_0) \subseteq X_0$, and that

(1) there is a compact S-invariant subset A_0 of X_0 which is globally asymptotically stable for S in X_0;

(2) Let A_∂ be the maximal compact invariant set of S in ∂X_0. $\tilde{A}_\partial = \underset{x \in A_\partial}{\cup} \omega_S(x)$ has an isolated and S-acyclic covering $\cup_{i=1}^k M_i$ in ∂X_0, that is, $\tilde{A}_\partial \subseteq \cup_{i=1}^k M_i$, where $M_1, M_2, \ldots, M_k$ are pairwise disjoint, compact and isolated invariant sets of S in ∂X_0 such that each M_i is also an isolated S-invariant set in X, and no subset of M_i's forms a cycle for $S_\partial = S|_{A_\partial}$ in A_∂;

(3) $\widetilde{W}^s(M_i) \cap X_0 = \emptyset$, $i = 1, 2, \cdots, k$, where $\widetilde{W}^s(M_i) = \{x; x \in X, \text{the } \omega - (T_n)\ (n \geq 0) - \text{limit set } \omega(x) \neq \emptyset \text{ and } \omega(x) \subseteq M_i\}$.

Then for any precompact orbit $\gamma^+(x)$ of $T_n, n \geq 0$, with $x \in X_0$, its ω-limit set $\omega(x) \subseteq A_0$.

Proof. Let $W^s(A_0)$ be the stable set of A_0 for S, then, clearly, $X_0 \subseteq W^s(A_0)$. By Theorem 2.3, it suffices to prove that $\omega(x) \cap X_0 \neq \emptyset$. Assume that, by contradiction, $\omega(x) \cap X_0 = \emptyset$, then $\omega(x) \subseteq \partial X_0$, and hence, since $\omega(x)$ is a compact S-invariant set, $\omega(x) \subseteq A_\partial$. Then $\cup_{y \in \omega(x)} \omega_S(y) \subseteq \cup_{y \in A_\partial} \omega_S(y) \subseteq \cup_{i=1}^k M_i$. Since $\omega_S(y) \subseteq \omega(x)$ for any $y \in \omega(x)$, $\omega(x) \cap \cup_{i=1}^k M_i \neq \emptyset$, and hence there exists some M_{i_1} $(1 \leq i_1 \leq k)$ such that $\omega(x) \cap M_{i_1} \neq \emptyset$. By assumption (3), $\omega(x) \nsubseteq M_i$ for all $i = 1, 2, \cdots, k$. By Theorem 2.2, there exists $w_1 \in \omega(x) \setminus M_{i_1}$ and a full S-orbit $\gamma_S(w_1) \subseteq \omega(x)$ and $\alpha_S(w_1) \subseteq M_{i_1}$. Since $w_1 \in \omega(x)$, $\omega_S(w_1) \subseteq \cup_{i=1}^k M_i$, and hence, by the compact S-invariant connectedness of $\omega_S(w_1)$, there exists some M_{i_2} $(1 \leq i_2 \leq k)$ such that $\omega_S(w_1) \subseteq M_{i_2}$. Therefore M_{i_1} is chained to M_{i_2}, i.e., $M_{i_1} \rightarrow M_{i_2}$. Clearly, $\omega_S(w_1) \subseteq \omega(x)$. Then $\omega(x) \cap M_{i_2} \neq \emptyset$ and, by assumption (3), $\omega(x) \nsubseteq M_{i_2}$. Again by Theorem 2.2, there exists $w_2 \in \omega(x) \setminus M_{i_2}$ and a full S-orbit $\gamma_S(w_2) \subseteq \omega(x)$ and $\alpha_S(w_2) \subseteq M_{i_2}$. We can repeat the above argument to get an i_3 $(1 \leq i_3 \leq k)$ such that $M_{i_2} \rightarrow M_{i_3}$. Since there are only a finite number of M_i's, we will eventually arrive at a cyclic chain of some M_i for S in A_∂, which contradicts our assumption (2).

This completes the proof.

Remark 2.1 If we assume that$S : X \rightarrow X$ is point dissipative, that S is compact, or alternatively, S is asymptotically smooth and $\gamma_S^+(U)$ is strongly bounded in X_0 if U is strongly bounded in X_0, and that S is uniformly persistent with respect to $(X_0, \partial X_0)$, then S admits a global attractor A_0 in X_0 which is globally asymptotically stable for S(see,e.g., [23,Theorem 2.1]).

§3. Asymptotically Periodic Semiflows

Let (X, d) be a metric space. A continuous mapping $\Phi : \Delta \times X \to X,\ \Delta = \{(t,s); 0 \leq s \leq t < \infty\}$, is called a nonautonomous semiflow if Φ satisfies the following properties:
(i) $\Phi(s, s, x) = x$, for all $s \geq 0,\ x \in X$;
(ii) $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x)$, for all $t \geq s \geq r \geq 0$.

Recall that $T(t) : X \to X,\ t \geq 0$ is called an ω-periodic semiflow on X if there is an $\omega > 0$ such that $T(t)x$ is continuous in $(t, x) \in [0, \infty) \times X,\ T(0) = I$, and $T(t+\omega) = T(t)T(\omega)$ for all $t \geq 0$ (see[9,23]). For convenience, we also use the notation $T(t, x) = T(t)x,\ x \in X, t \geq 0$.

Definition 3.1 A nonautonomous semiflow $\Phi : \Delta \times X \to X$ is called asymptotically periodic-with limit periodic semiflow $T(t) : X \to X, t \geq 0$, if

$$\Phi(t_j + n_j\omega, n_j\omega, x_j) \to T(t)x,\ j \to \infty,$$

for any three sequences $t_j \to t, n_j \to \infty, x_j \to x$ $(j \to \infty)$ with $x, x_j \in X$.

For an asymptotically periodic semiflow $\Phi : \Delta \times X \to X$ with limit ω-periodic semiflow $T(t) : X \to X, t \geq 0$, let $T_n(x) = \Phi(n\omega, 0, x),\ n \in N, x \in X$, and $S = T(\omega) : X \to X$. Define $S_n : X \to X, n \geq 0$, by $S_n(x) = \Phi((n+1)\omega, n\omega, x), n \geq 0, x \in X$. Then, by the properties of nonautonomous semiflows,$T_n(x) = S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0,\ n \geq 1, x \in X$. By Definition 3.1, it then easily follows that $\lim_{(n,x)\to(\infty,x_0)} S_n(x) = S(x_0)$, i.e., $T_n : X \to X,\ n \geq 0$, is an asymptotically autonomous discrete dynamical process with limit autonomous discrete semiflow $S^n : X \to X, n \geq 0$, in the sense of Definition 2.1.

We are now in a position to prove the main result of this section.

Theorem 3.1 Let $\Phi : \Delta \times X \to X$ be an asymptotically periodic semiflow with limit ω-periodic semiflow $T(t) : X \to X, t \geq 0$, and $T_n(x) = \Phi(n\omega, 0, x), n \geq 0, x \in X$ and $S(x) = T(\omega)x, x \in X$. Assume that A_0 is a compact S-invariant subset of X. If for some $y \in X$, $\lim_{n\to\infty} d(T_n(y), A_0) = 0$, then

$$\lim_{t\to\infty} d(\Phi(t,0,y), T(t)A_0) = 0.$$

Proof. We first prove the following claim.

Claim. $\lim_{(n,x)\to(\infty,A_0)} d(\Phi(t+n\omega, n\omega, x), T(t)A_0) = 0$ uniformly for $t \in [0,\omega]$, that is, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ and $N = N(\epsilon) > 0$ such that for any $x \in N(A_0, \delta)$, $n \geq N$, and all $t \in [0,\omega]$, $\Phi(t+n\omega, n\omega, x) \in N(T(t)A_0, \epsilon)$, where $N(A_0,\delta) = \{x; d(x, A_0) < \delta\}$ is the δ-neighborhood of A_0.

Let $x_0 \in X$ be given. For any $\epsilon > 0$, since $T(t, x_0)$ is uniformly continuous for t in the compact set $[0,\omega]$, there exists $\delta_0 = \delta_0(\epsilon) > 0$, such that for any $t_1, t_2 \in [0,\omega]$ with $|t_1 - t_2| < \delta_0$,

$$\|T(t_1, x_0) - T(t_2, x_0)\| < \epsilon/2.$$

For any $t_0 \in [0,\omega]$, since, by Definition 3.1, $\lim_{(t,n,x)\to(t_0,\infty,x_0)} \Phi(t+n\omega, n\omega, x) = T(t_0, x_0)$, there exist $\delta = \delta(t_0,\epsilon) \leq \delta_0$, and $N = N(t_0, \epsilon) > 0$, such that for any $|t - t_0| < \delta$, $n \geq N$, and $x \in N(x_0, \delta)$,

$$\|\Phi(t+n\omega, n\omega, x) - T(t_0, x_0)\| < \epsilon/2.$$

Let $I(t_0,\delta) = (t_0 - \delta, t_0 + \delta)$. Since $\cup_{t_0\in[0,\omega]} I(t_0,\delta) \supseteq [0,\omega]$, the compactness of $[0,\omega]$ implies that there exist $t_1, t_2, \cdots, t_m \in [0,\omega]$ such that $\cup_{i=1}^m I(x_i, \delta_i) \supseteq [0,\omega]$. Let $N^* = \max_{1\leq i\leq m}\{N(t_i,\epsilon)\}$, $\delta^* = \min_{1\leq i\leq m}\{\delta_i = \delta(t_i,\epsilon)\}$. Then for any $x \in N(x_0, \delta^*)$, any $n \geq N^*$, and all $t \in [0,\omega]$, there exists some i $(1 \leq i \leq m)$ such that $t \in I(t_i, \delta_i)$. Therefore for all $n \geq N^* \geq N_i$, $\|x - x_0\| < \delta^* \leq \delta_i$, and $|t - t_i| < \delta_i \leq \delta_0$, and hence

$$\begin{aligned}&\|\Phi(t+n\omega, n\omega, x) - T(t, x_0)\|\\ &\leq \|\Phi(t+n\omega, n\omega, x) - T(t_i, x_0)\| + \|T(t_i, x_0) - T(t, x_0)\|\\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Therefore

$$\lim_{(n,x)\to(\infty,x_0)} (\Phi(t+n\omega, n\omega, x) - T(t,x_0)) = 0, \quad \text{uniformly for } t \in [0,\omega].$$

For $\epsilon > 0$ and any $x_0 \in A_0$, there exist $\delta = \delta(\epsilon, x_0) > 0$ and $N = N(\epsilon, x_0) > 0$ such that for any $x \in N(x_0, \delta),\ n \geq N$ and all $t \in [0, \omega]$,

$$\Phi(t + n\omega, n\omega, x) \in N(T(t)x_0, \epsilon).$$

Since $A_0 \subseteq \cup_{x_0 \in A_0} N(x_0, \delta/2)$, by the compactness of A_0, there exist $x_1, x_2, \cdots, x_k \in A_0$ such that $A_0 \subseteq \cup_{i=1}^k N(x_i, \delta_i/2)$. Let $\delta^* = \min_{1 \leq i \leq k}\{\delta_i/2\}$. For any $z \in N(A_0, \delta^*)$, there exists $x \in A_0$, such that $d(x, z) < \delta^*$, then there exists $x_i,\ (1 \leq i \leq k)$ such that $x \in N(x_i, \delta_i/2)$, and hence

$$d(z, x_i) \leq d(x, z) + d(x, x_i) < \delta^* + \delta_i/2 \leq \delta_i/2 + \delta_i/2 = \delta_i,$$

i.e., $z \in N(x_i, \delta_i)$. Then $N(A_0, \delta^*) \subseteq \cup_{i=1}^k N(x_i, \delta_i)$. Therefore for all $x \in N(A_0, \delta^*),\ n \geq N^* = \max_{1 \leq i \leq k}\{N(\epsilon, x_i)\}$, and $t \in [0, \omega]$, there exists some $x_i,\ (1 \leq i \leq k)$ such that $x \in N(x_i, \delta_i)$, and hence $n \geq N^* \geq N_i(\epsilon, x_i)$. Therefore, for all $t \in [0, \omega]$,

$$\Phi(t + n\omega, n\omega, x) \in N(T(t)x_i, \epsilon),$$

which implies $d(\Phi(t + n\omega, n\omega, x), T(t)A_0) < \epsilon$, that is,

$$\lim_{(n,x) \to (\infty, A_0)} d(\Phi(t + n\omega, n\omega, x), T(t)A_0) = 0, \quad \text{uniformly for } t \in [0, \omega].$$

For any $t \geq 0$, let $t = n\omega + t'$, where $n = [t/\omega]$ is the greatest integer less than or equal to t/ω and $t' \in [0, \omega)$, then $\Phi(t, 0, y) = \Phi(t, n\omega, \Phi(n\omega, 0, y))$, and by the S-invariance of A_0, $T(t)A_0 = T(t')T(n\omega)A_0 = T(t')A_0$. Therefore, since $\lim_{n \to \infty} d(\Phi(n\omega, 0, y), A_0) = \lim_{n \to \infty} d(T_n(y), A_0) = 0$, and by the claim above,

$$\begin{aligned} &\lim_{t \to \infty} d(\Phi(t, 0, y), T(t)A_0) \\ &= \lim_{t \to \infty} d(\Phi(t' + n\omega, n\omega, \Phi(n\omega, 0, y)), T(t')A_0) = 0. \end{aligned}$$

This completes the proof.

Now we turn to the concrete examples of asymptotically periodic semiflows generated by systems of parabolic equations and ordinary differential equations under some appropriate conditions.

Consider first the systems of parabolic differential equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + A_i(t)u_i = f_i(x,t,u_1,\cdots,u_m) & \text{in } \ \Omega \times (0,\infty) \\ \qquad\qquad B_i u_i = 0 & \text{on } \ \partial\Omega \times (0,\infty) \end{cases} \tag{3.1}$$

where $i = 1, \cdots, m$, and $\Omega \subset R^N$ $(N \geq 1)$ is a bounded domain with boundary $\partial\Omega$ of class $C^{2+\theta}$ $(0 < \theta \leq 1)$,

$$A_i(t)v = -\sum_{j,k=1}^{N} a_{jk}^{(i)}(x,t)\frac{\partial^2 v}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j^{(i)}(x,t)\frac{\partial v}{\partial x_j} + a_0^{(i)}(x,t)v$$

$(1 \leq i \leq m)$ are linear uniformly elliptic differential expressions of second order for each $t \in [0,\omega]$ $(\omega > 0)$ and $A_i(t)$ are ω-periodic in t, and $B_i v = v$ or $B_i v = \frac{\partial v}{\partial n} + b_0^{(i)}(x)v$, where $\frac{\partial}{\partial n}$ denotes the differentiation in the direction of the outward normal n to $\partial\Omega$. We assume that $a_{jk}^{(i)} = a_{kj}^{(i)}$, $a_j^{(i)}$ and $a_0^{(i)} \in C^{\theta,\theta/2}(\overline{Q}_\omega)$, $a_0^{(i)} \geq 0$ $(1 \leq j,k \leq N, 1 \leq i \leq m)$, $Q_\omega = \Omega \times [0,\omega]$, and $b_0^{(i)} \in C^{1+\theta}(\partial\Omega, R)$, $b_0^{(i)} \geq 0$ $(i = 1, 2, \cdots, m)$.

We further impose the following smoothness condition on $f = (f_1, \cdots, f_m)^T$.

(H) $f_i \in C(\overline{\Omega} \times R_+ \times R^m, R)$, $\frac{\partial f_i}{\partial u_j}$ exists and $\frac{\partial f_i}{\partial u_j} \in C(\overline{\Omega} \times R_+ \times R^m, R)$, and for any $T > 0$, $f_i(\cdot,\cdot,u)$ and $\frac{\partial f_i}{\partial u_j}(\cdot,\cdot,u) \in C^{\theta,\theta/2}(\overline{Q}_T, R)$ uniformly for $u = (u_1, \cdots, u_m)$ in bounded subsets of R^m $(i, j = 1, \cdots, m)$.

Let $X = L^p(\Omega)$, $N < p < \infty$, and for $\beta \in (\frac{1}{2} + \frac{N}{2p}, 1)$, let $E_i = X_\beta^{(i)}$ $(i = 1, \cdots, m)$ be the fractional power space of X with respect to $(A_i(0), B_i)$ (e.g., see Henry [11]), then E_i is an ordered Banach space with the order cone P_i consisting of all nonnegative functions in E_i and P_i has nonempty interior $int(P_i)$. Let $E = E_1 \times \cdots \times E_m$, then by an easy extension of some results in [12, Section III.20] to the systems, it follows that for every $u = (u_1, \cdots, u_m) \in E$ and every $s \geq 0$, there exists a unique regular solution $\phi(t,s,u)$ of (3.1) satisfying $\phi(s,s,u) = u$ with its maximal existence interval $I^+(s,u) \subset [s,\infty)$ and $\phi(t,s,u)$ is globally defined provided there is an L^∞-bound on $I^+(s,u)$.

Now assume that f_i^0, $i = 1, 2, \cdots, m$, are ω-periodic in t and satisfy (H).For any $u \in E$, let $\phi_0(t,s,u)$ be the unique solution of the following ω-periodic systems of parabolic equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + A_i(t)u_i = f_i^0(x,t,u_1,\cdots,u_m) & \text{in } \ \Omega \times (0,\infty) \\ \qquad\qquad B_i u_i = 0 & \text{on } \ \partial\Omega \times (0,\infty) \end{cases} \tag{3.2}$$

with $\phi_0(s,s,u) = u$, and let $T(t,u) = \phi_0(t,0,u)$. Then we have the following result.

Proposition 3.1 Let $f = (f_1,\cdots,f_m)^T,\ f_0 = (f_1^0,\cdots,f_m^0)^T,\ \|u\|_E = \sum_{i=1}^m \|u_i\|_{E_i}$ for $u \in E$, and $|u| = \sum_{i=1}^m |u_i|$ for $u \in R^m$. Assume that

(1) $\lim_{t\to\infty} |f(x,t,u) - f_0(x,t,u)| = 0$ uniformly for $x \in \overline{\Omega}$ and u in any bounded set of R^m;

(2) solutions of (3.1) and (3.2) are uniformly bounded in E, i.e., for any $r > 0$, there exists $B = B(r) > 0$ such that for any $u \in E$ with $\|u\| \le r$, $\|\phi(t,s,u)\| \le B(r)$, and $\|\phi_0(t,s,u)\| \le B(r),\quad t \ge s \ge 0$.

Then for any given $k \in N$ $(k > 0)$, and $r > 0$

$$\lim_{n\to\infty} \|\phi(t+n\omega, n\omega, u) - T(t,u)\|_E = 0$$

uniformly for $t \in [0, k\omega]$ and $\|u\| \le r$. In particular, for any $u \in E$, $\gamma^+(u) = \{\phi(n\omega,0,u); n \ge 0\}$ is precompact in E, and $\phi : \Delta \times E \to E$ is an asymptotically periodic semiflow with limit periodic semiflow $T(t) : E \to E, t \ge 0$.

Proof. For any $u \in E$, by the uniform boundedness, $\phi(t,s,u)$ and $\phi_0(t,s,u)$ exist globally on $[s,\infty)$ for any $s \ge 0$. Given $r > 0$, let $B = B(r)$ be as in assumption (2), then there exists $B_1 = B_1(B) > 0$ such that $\|\phi((t,s,u)\|_{C(\overline{\Omega})} \le B_1,\ \|\phi_0((t,s,u)\|_{C(\overline{\Omega})} \le B_1$, for all $t \ge s \ge 0$ and $u \in E$ with $\|u\| \le r$. Let

$$\phi(t, n\omega, u) = \tilde{u}(t) = (\tilde{u}_1(t),\cdots,\tilde{u}_m(t)),\ t \ge n\omega,\ n \ge 0,$$

and

$$\phi_0(t, n\omega, u) = u(t) = (u_1(t),\cdots,u_m(t)),\ t \ge n\omega, n \ge 0.$$

Let $U_i(t,\tau)$ be the evolution operator generated by $A_i(t),\ 1 \le i \le n$ (see [12, II.11]). Then by the variation of constant's formula(see, e.g., [12, III.19]),

$$\tilde{u_i}(t) = U_i(t,n\omega)u_i + \int_{n\omega}^t U_i(t,s) f_i(\cdot, s, \tilde{u}(s))ds,$$

and

$$u_i(t) = U_i(t, n\omega)u_i + \int_{n\omega}^{t} U_i(t,s) f_i^0(\cdot, s, u(s))ds,$$

for all $n\omega \leq t \leq (n+k)\omega$. Let $D_m = \overline{\Omega} \times [n\omega, (n+k)\omega] \times [0, B_1]^m \subset R^N \times R_+ \times R^m$. Then

$$\begin{aligned}
\|\tilde{u}_i(t) - u_i(t)\|_\beta \leq & \int_{n\omega}^{t} \|U_i(t,s)\|_{0,\beta} \cdot \left\|f_i^0(\cdot, s, \tilde{u}(s)) - f_i^0(\cdot, s, u(s))\right\|_0 ds \\
& + \int_{n\omega}^{t} \|U_i(t,s)\|_{0,\beta} \cdot \left\|f_i(\cdot, s, \tilde{u}(s)) - f_i^0(\cdot, s, \tilde{u}(s))\right\|_0 ds \\
& \leq c_0 \int_{n\omega}^{t} \|U_i(t,s)\|_{0,\beta} \cdot \|\tilde{u}(s)) - u(s))\|_\beta \, ds \\
& + \int_{n\omega}^{t} \|U_i(t,s)\|_{0,\beta} \cdot \left\|f_i - f_i^0\right\|_{C(D_m, R)} ds.
\end{aligned}$$

For a fixed $\alpha \in (\beta, 1)$, using the estimates (see[12, II.11])

$$\|U_i(t,s)\|_{0,\beta} \leq c_i(t-s)^{-\alpha}, \text{and} \int_{n\omega}^{t} (t-s)^{-\alpha} ds \leq \frac{(k\omega)^{1-\alpha}}{1-\alpha},$$

for all $n\omega \leq s \leq t \leq (n+k)\omega$, we have

$$\begin{aligned}
&\|\tilde{u}(t) - u(t)\|_E = \sum_{i=1}^{m} \|\tilde{u}_i(t) - u_i(t)\|_\beta \\
&\leq c \int_{n\omega}^{t} (t-s)^{-\alpha} \|\tilde{u}(s) - u(s)\|_\beta \, ds + c \int_{n\omega}^{t} (t-s)^{-\alpha} \|f - f_0\|_{C(D_m, R^m)} ds,
\end{aligned}$$

where $c = c(k, r) > 0$, and hence by a version of Gronwall's inequality(see, e.g., [12, Lemma 19.4])

$$\begin{aligned}
\|\phi(t, n\omega, u) - \phi_0(t, n\omega, u)\|_E =& \|\tilde{u}(t) - u(t)\|_E \\
\leq& \bar{c} \|f - f_0\|_{C(D_m, R^m)},
\end{aligned}$$

for all $t \in [n\omega, (n+k)\omega]$ and $\|u\|_E \leq r$. Since (3.2) is ω-periodic system, $\phi_0(n\omega + t, n\omega, u) = \phi_0(t, 0, u) = T(t, u)$. Therefore for any $t \in [0, k\omega]$ and $\|u\| \leq r$,

$$\begin{aligned}
\|\phi(n\omega + t, n\omega, u) - T(t,u)\|_\beta =& \|\phi(n\omega + t, n\omega, u) - \phi_0(n\omega + t, n\omega, u)\|_\beta \\
\leq& \bar{c} \|f - f_0\|_{C(D_m, R^m)}.
\end{aligned}$$

It then follows that

$$\lim_{n\to\infty}(\phi(n\omega+t,n\omega,u)-T(t,u))=0, \text{ uniformly for } t\in[0,k\omega] \text{ and } \|u\|\le r. \tag{3.3}$$

For any $u\in E$, let $T_n(u)=\phi(n\omega,0,u)$, $S_n(u)=\phi((n+1)\omega,n\omega,u)$ and $S(u)=T(\omega,u)$, then, by (3.3),

$$\lim_{n\to\infty}\|S_n(u)-S(u)\|_E=0, \quad \text{uniformly for } \|u\|\le r\ (r>0).$$

For any $u\in E$, there exists $r>0$ such that $\|\phi(t,s,u)\|_E\le r,\ t\ge s\ge 0$ (by the uniform boundedness of (3.1)). Then $\|T_n(u)\|_E=\|\phi(n\omega,0,u)\|\le r, n\ge 0$, and hence

$$\lim_{n\to\infty}\|T_{n+1}(u)-S(T_n(u))\|_E=\lim_{n\to\infty}\|S_n(T_n(u))-S(T_n(u))\|_E=0. \tag{3.4}$$

Since S is the Poincaré map of periodic parabolic system (3.2), $S:E\to E$ is continuous and compact(see,e.g., [12,III.21]). Then $S(\gamma^+(u))$ is precompact in E, and hence, (3.4) implies that $\gamma^+(u)=\{T_n(u);n\ge 0\}$ is precompact in E.

For any $(t_0,u_0)\in R_+\times E$, let $k\in N$ $(k>0)$ and $r>0$ be such that $t_0\in[0,k\omega]$ and $\|u_0\|<r$. For any $t\in[0,k\omega]$ and $\|u\|\le r$,
$\|\phi(t+n\omega,n\omega,u)-T(t_0,u_0)\|_E$
$\le\|\phi(t+n\omega,n\omega,u)-T(t,u)\|_E+\|T(t,u)-T(t_0,u_0)\|_E$.
By (3.3) and the continuity of $T(t,u)$ for $(t,u)\in R_+\times E$, it follows that

$$\lim_{(t,u,n)\to(t_0,u_0,\infty)}\|\phi(t+n\omega,n\omega,u)-T(t_0,u_0)\|_E=0,$$

i.e., $\phi(t,s,u):\Delta\times E\to E$ is asymptotic to ω-periodic semiflow $T(t):E\to E, t\ge 0$.

This completes the proof.

We then consider the systems of ordinary differential equations

$$\frac{du}{dt}=f(u,t), \quad u\in R^m \tag{3.5}$$

and

$$\frac{du}{dt}=f_0(u,t), \quad u\in R^m \tag{3.6}$$

Assume that $f(u,t) : R^m \times R_+ \to R^m$ is continuous and locally Lipschitz in u, and that $f_0(u,t) : R^m \times R_+ \to R^m$ is continuous, ω-periodic in t and locally Lipschitz in u uniformly for $t \in [0,\omega]$. Let $\phi(t,s,u)$ and $\phi_0(t,s,u)$ be the unique solutions of (3.5) and (3.6) with $\phi(s,s,u) = u$ and $\phi_0(s,s,u) = u$ $(s \geq 0)$, repectively, and let $T(t,u) = \phi_0(t,0,u),\ t \geq 0$.

By a similar Gronwall's inequality argument as in Proposition 3.1, we can prove the following result.

Proposition 3.2 Assume that

(1) $\lim_{t\to\infty} |f(u,t) - f_0(u,t)| = 0$ uniformly for u in any bounded subset of R^m;

(2) solutions of (3.5) and (3.6) are uniformly bounded in R^m.

Then for any $k \in N$ $(k > 0)$ and $r > 0$,

$$\lim_{n\to\infty} |\phi(t + n\omega, n\omega, u) - T(t,u)| = 0$$

uniformly for $t \in [0, k\omega]$ and $|u| \leq r$, and in particular, $\phi : \Delta \times R^m \to R^m$ is asymptotic to the ω-periodic semiflow $T(t) : R^m \to R^m, t \geq 0$.

Remark 3.1 By theorem 3.1, we can reduce the study of asymptotic behavior of an asymptotically periodic semiflow $\Phi : \Delta \times X \to X$ with limit ω-periodic semiflow $T(t) : X \to X, t \geq 0$ to that of its associated asymptotically autonomous discrete dynamical process $T_n : X \to X, n \geq 0$ with limit autonomous discrete semiflow $S^n : X \to X, n \geq 0$, where $S = T(\omega) : X \to X$ is the usual Poincaré map of the ω-periodic semiflow $T(t) : X \to X, t \geq 0$. Accordingly, the general results in Section 2 can be applied . As an illustration, we will discuss some asymptotically periodic parabolic equations and systems in the next section.

§4. Some Applications

In this section, we will apply some general results in Sections 2 and 3 to discuss the global asymptotic behavior of asymptotically periodic parabolic Kolmogorov equations and predator-prey systems.

Consider systems of parabolic equations

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + A_i(t)u_i = u_i G_i(x, t, u_1, \cdots, u_m) & \text{in } \ \Omega \times (0, \infty) \\ \qquad\qquad B_i u_i = 0 & \text{on } \ \partial\Omega \times (0, \infty) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + A_i(t)u_i = u_i G_i^0(x, t, u_1, \cdots, u_m) & \text{in } \ \Omega \times (0, \infty) \\ \qquad\qquad B_i u_i = 0 & \text{on } \ \partial\Omega \times (0, \infty) \end{cases} \tag{4.2}$$

where $1 \leq i \leq m$, and $A_i(t), B_i$ and Ω are as in Section 3. We assume that $G_i^0, 1 \leq i \leq m$ are ω-periodic in t for some $\omega > 0$, and that G_i and $G_i^0, 1 \leq i \leq m$, satisfy the smoothness condition (H). For any $u \in E = E_1 \times \cdots \times E_m$, let $\phi(t, s, u)$ and $\phi_0(t, s, u)$ $(s \geq 0)$ be the unique solutions of (4.1) and (4.2) with $\phi(s, s, u) = u$ and $\phi_0(s, s, u) = u$, respectively. Moreover, by an invariant principle argument (see, e.g., [1,18]), it follows that any solutions $\phi(t, s, u)$ and $\phi_0(t.s, u)$ of (4.1) and (4.2) with nonnegative initial values remains nonnegative.

For any $m \in C^{\theta,\theta/2}(\overline{Q}_\omega)$, according to [12], there exists a unique principal eigenvalue of the periodic-parabolic eigenvalue problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + A_i(t)v = m(x, t)v + \mu v & \text{in } \quad \Omega \times R \\ B_i v = 0 & \text{on } \partial\Omega \times R \\ v \ \ \omega - \text{periodic in } t \ \ , \end{cases}$$

which we denote by $\mu^{(i)}(m(x, t)), \ i = 1, \cdots, m$.

We need the following result in the application of Theorems 2.3 and 2.5 to asymptotically periodic parabolic systems.

Proposition 4.1 Assume that conditions (1) and (2) of Proposition 3.1 with $f_i = u_i G_i(x, t, u)$ and $f_i^0 = u_i G_i^0(x, t, u), \ 1 \leq i \leq m$, hold. Let $u^*(t) = (u_1^*(t), \cdots, u_{i-1}^*(t), 0, u_{i+1}^*(t), \cdots, u_m^*(t))$ be a nonnegative ω-periodic solution of (4.2) for some $1 \leq i \leq m$. If $\mu^{(i)}(G_i^0(x, t, u^*(t)) < 0$, then

$$\widetilde{W}^s(u^*(0)) \cap X_0 = \emptyset,$$

where $X_0 = \{u \in E; u_i(x) \geq 0 \text{ and } u_i(x) \not\equiv 0, x \in \Omega, 1 \leq i \leq m\}$, and $\widetilde{W}^s(u^*(0))$ is the stable set of $u^*(0)$ with respect to $T_n = \phi(n\omega, 0, \cdot) : X \to X, n \geq 0$.

Proof. Assume that, by contradiction, there exists a $u_0 \in X_0 \cap \widetilde{W}^s(u^*(0))$, i.e., $u_0 \in X_0$, and $T_n(u_0) \to u^*(0)$ as $n \to \infty$. Then $u(t) = \phi(t, 0, u_0), t \geq 0$,satisfies $u(t) \gg 0$ for all $t > 0$, and, by Theorem 3.1, $\lim_{t\to\infty} \|u(t) - u^*(t)\|_E = 0$, and hence $\lim_{t\to\infty} \|u(t) - u^*(t)\|_{C(\overline{\Omega})} = 0$. Then there exists $M > 0$ such that $\|u(t)\|_{C(\overline{\Omega})} \leq M$ and $\|u^*(t)\|_{C(\overline{\Omega})} \leq M$, for all $t \geq 0$. Since for all $x \in \Omega$ and $t \geq 0$,

$$\begin{aligned}|G_i(x, t, u(t)) - G_i^0(x, t, u^*(t))| \leq & |G_i(x, t, u(t)) - G_i^0(x, t, u(t))| \\ & + |G_i^0(x, t, u(t)) - G_i^0(x, t, u^*(t))|,\end{aligned}$$

$\lim_{t\to\infty} |G_i(x, t, u(t)) - G_i^0(x, t, u^*(t))|_{C(\overline{\Omega})} = 0$. Choose $0 < \epsilon \ll 1$ such that $\mu_\epsilon^{(i)} = \mu^{(i)}(G_i^0(x, t, u^*(t)) - \epsilon) < 0$(by [12, Lemma 15.7]). Then there exists $N = N(\epsilon) > 0$ such that $G_i(x, t, u(t)) \geq G_i^0(x, t, u^*(t)) - \epsilon$ for all $x \in \Omega$ and $t \geq N\omega$. Therefore $u_i(t, x)$ satisfies

$$\begin{aligned}\frac{\partial u_i}{\partial t} + A_i(t)u_i &\geq u_i\left(G_i^0(x, t, u^*(t)) - \epsilon\right) \\ &> u_i\left(F_i^0(x, t, u^*(t)) - \epsilon\right) + \mu_\epsilon^{(i)} u_i,\end{aligned}$$

for all $x \in \Omega$ and $t \geq N\omega$. Let $\varphi_i \gg 0$ be the principal eigenfunction corresponding to $\mu_\epsilon^{(i)}$, that is, φ_i satisfies

$$\begin{cases} \dfrac{\partial \varphi_i}{\partial t} + A_i(t)\varphi_i = \varphi_i(G_i^0(x, t, u^*(t)) - \epsilon) + \mu_\epsilon^{(i)}\varphi_i & \text{in } \Omega \times (0, \infty) \\ B_i\varphi_i = 0 & \text{on } \partial\Omega \times (0, \infty) \\ \varphi_i \quad \omega - \text{periodic in } t. \end{cases}$$

Since $u_i(N\omega) \gg 0 \quad \text{in } E_i$, there exists $k = k(\epsilon, u_0) > 0$ such that $u_i(N\omega) \geq k\varphi_i(N\omega, \cdot) = k\varphi(0, \cdot)$. Then by comparison theorem,

$$u_i(t) \geq k\varphi_i(t, \cdot), \quad \text{for all } t \geq N\omega.$$

In particular, $u_i(n\omega) \geq k\varphi(0, \cdot), \quad$ for all $n \geq N$, which contradicts that $\lim_{n\to\infty} u_i(n\omega) = 0 \ \text{ in } E_i$.

This completes the proof.

Now we consider the following scalar asymptotically periodic parabolic Kolmogorov equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(t)u = uF(x,t,u) & \text{in } \Omega \times (0,\infty) \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{4.3}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(t)u = uF_0(x,t,u) & \text{in } \Omega \times (0,\infty) \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty) \end{cases} \tag{4.4}$$

where $A(t), B$ and Ω satisfy the same conditions as A_i, B_i and Ω in (4.1). We assume that F_0 is ω-periodic for some $\omega > 0$, and that F and F_0 satisfy the smoothness condition (H). We further assume that

(C1) $\lim_{t\to\infty} |F(x,t,u) - F_0(x,t,u)| = 0$ uniformly for $x \in \overline{\Omega}$ and u in any bounded subset of R^+, and there exists $K > 0$ such that $F(x,t,u) \leq 0$ for all $(x,t) \in \overline{\Omega} \times R_+$ and $u \geq K$;

(C2) For any $(x,t) \in \overline{Q}_\omega$, $F_0(x,t,u)$ is nonincreasing for u and for at least one $(x_0,t_0) \in Q_\omega$, $F_0(x_0,t_0,u)$ is strictly nonincreasing for u , and there exists $K_0 > 0$ such that $F_0(x,t,K_0) \leq 0$ for all $(x,t) \in \overline{Q}_\omega$.

Let $X = L^p(\Omega)$, $N < p < \infty$, and for $\beta \in (\frac{1}{2} + \frac{N}{2p}, 1)$, let X_β be the fractional power space of X with respect to $(A(0), B)$, then X_β is an ordered Banach space with the order cone X_β^+ consisting of all nonnegative functions in X_β. For any $u \in X_\beta^+$ and $s \geq 0$, let $\phi(t,s,u)$ and $\phi_0(t,s,u)$ be the unique solutions of (4.3) and (4.4) with $\phi(s,s,u) = u$ and $\phi_0(s,s,u) = u$, respectively. Then we have the following threshold type result.

Theorem 4.1 Assume that (C1) and (C2) hold.
(a) If $\mu(A(t), F_0(x,t,0)) \geq 0$, then for any $u_0 \in X_\beta^+$, $\lim_{t\to\infty} \phi(t,0,u_0) = 0$ in X_β;
(b) If $\mu(A(t), F_0(x,t,0)) < 0$, then for any $u_0 \in X_\beta^+ \setminus \{0\}$, $\lim_{t\to\infty} \|\phi(t,0,u_0) - u^*(t)\| = 0$ in X_β, where $u^*(t)$ is the unique positive ω-periodic solution of (4.4).

Proof. By conditions (C1) and (C2), it easily follows that for any $s \geq 0$, $\phi(t,s,u)$ and $\phi_0(t,s,u)$ exist globally on $[s,\infty)$ and are uniformly bounded in X_β^+. Then, by Proposition 3.1, $\phi(t,s,u)$ is asymptotic to an ω-periodic semiflow $T(t)u = \phi_0(t,0,u),\ t \geq 0$, in X_β^+, and for any $u \in X_\beta^+$, $\gamma^+(u) = \{T_n(u); n \geq 0\}$, where $T_n(u) = \phi(n\omega, 0, u), n \geq 0$, is precompact in X_β^+ and hence its ω-limit set $\omega(u)$ exists. By Theorem 3.1, it suffices to prove that $\lim_{n\to\infty} T_n(u) = 0$ for any $u \in X_\beta^+$ in case (a), and $\lim_{n\to\infty} T_n(u) = u^*(0)$ for any $u \in X_\beta^+ \setminus \{0\}$ in case (b), respectively. Notice that $T_n : X_\beta^+ \to X_\beta^+, n \geq 0$ is an asymptotically autonomous discrete dynamical process with limit discete semiflow $S^n : X_\beta^+ \to X_\beta^+, n \geq 0$, where $S = T(\omega)$ is the Poincaré map associated with periodic equation (4.4).

Case (a). By [22, Theorem 3.2], $u = 0$ is a globally asymptotically stable fixed point of S, and then $W^s(0) = X_\beta^+$, where $W^s(0)$ is the stable set of 0 for S in X_β^+. Clearly, for any $u \in X_\beta^+$, $\omega(u) \subseteq X_\beta^+$ and hence $\omega(u) \cap X_\beta^+ \neq \emptyset$. By Theorem 2.3, $\omega(u) = 0$, i.e., $\lim_{n\to\infty} T_n(u) = 0$.

Case (b). By [22,Theorem 3.3], $u = u^*(0)$ is a globally asymptotically stable fixed point of S in $X_\beta^+ \setminus \{0\}$, and hence $W^s(u^*(0)) = X_\beta^+ \setminus \{0\}$, where $W^s(u^*(0))$ is the stable set of $u^*(0)$ for S. Since $\mu(A(t), F_0(x,t,0)) < 0$, by Proposition 4.1, $\widetilde{W^s}(0) \cap (X_\beta^+ \setminus \{0\}) = \emptyset$. Then, for any $u \in X_\beta^+ \setminus \{0\}$, $\omega(u) \cap (X_\beta^+ \setminus \{0\}) \neq \emptyset$, i.e., $\omega(u) \cap W^s(u^*(0)) \neq \emptyset$. Therefore, by Theorem 2.3, for any $u \in X_\beta^+ \setminus \{0\}$, $\omega(u) = u^*(0)$, i.e., $\lim_{n\to\infty} T_n(u) = u^*(0)$ in X_β.

This completes the proof.

Remark 4.1 By the similar argument as in Theorem 4.1, one can easily prove the main result([13, Theorem A]) on the asymptotically periodic parabolic Fisher equations. Notice that in the case (1) of [13, Theorem A], since every fixed point of S is stable, it is easy to prove that the set of all fixed points of the Poincaré map S of the associated periodic parabolic Fisher equation is a globally asymptotically stable compact and S-invariant set for S in $X = [0,1]_E$, and then Theorem 2.3 applies.

Finally we consider the following asymptotically periodic parabolic predator-prey system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + A_i(t)u_i = u_i F_i(x,t,u_1,u_2) & \text{in } \ \Omega \times (0,\infty) \\ \qquad\qquad B_i u_i = 0 & \text{on } \ \partial\Omega \times (0,\infty) \end{cases} \tag{4.5}$$

and

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + A_i(t)u_i = u_i F_i^0(x,t,u_1,u_2) & \text{in } \ \Omega \times (0,\infty) \\ \qquad\qquad B_i u_i = 0 & \text{on } \ \partial\Omega \times (0,\infty) \end{cases} \tag{4.6}$$

where $1 \le i \le 2$, and $A_i(t), B_i$ and Ω are as in Section 3. We assume that $F_i^0, 1 \le i \le 2$, are ω-periodic in t for some $\omega > 0$, and that F_i and $F_i^0, 1 \le i \le 2$, satisfy the smoothness condition (H). Let $E_i = X_\beta^{(i)}, i = 1,2$, be as in Section 3, and let P_i be the positive cone of $E_i, i = 1,2$, respectively. For any $u \in E = E_1 \times E_2$, let $\phi(t,s,u)$ and $\phi_0(t,s,u)$ $(s \ge 0)$ be the unique solutions of (4.5) and (4.6) with $\phi(s,s,u) = u$ and $\phi_0(s,s,u) = u$, respectively. Let $F = (F_1, F_2)^T$ and $F_0 = (F_1^0, F_2^0)^T$. For predator-prey models, assume that prey u_1 and predator u_2 live in a bounded habitat Ω. We further make the following assumptions.

(M1) $\lim_{t\to\infty} |F(x,t,u) - F_0(x,t,u)| = 0$ uniformly for $x \in \overline{\Omega}$ and u in any bounded subset of R_+^2;

(M2) For any $(x,t,u_1,u_2) \in \overline{\Omega} \times R_+^3$, $F_1(x,t,u_1,u_2) \le F_1(x,t,u_1,0)$, and there exist $a_1 > 0$ and $M_1 > 0$ such that $F_1(x,t,u_1,0) \le -a_1 < 0$ for all $(x,t) \in \overline{\Omega} \times R_+$ and $u_1 \ge M_1$; For any given $(x,t,u_2) \in \overline{\Omega} \times R_+^2$, $F_2(x,t,u_1,u_2)$ is increasing for $u_1 \ge 0$, and for any $M > 0$, there exist $a_2(M) > 0$ and $M_2(M) > 0$ such that $F_2(x,t,M,u_2) \le -a_2 < 0$ for all $(x,t) \in \overline{\Omega} \times R_+$ and $u_2 \ge M_2$;

(M3) F_1^0 and F_2^0 satisfy similar conditions to (M2);

(M4) For any given $(x,t) \in \overline{Q}_\omega$, $F_1^0(x,t,u_1,0)$ is strictly decreasing in $u_1 \in R_+$, and there exists $M > 0$ such that $F_1^0(x,t,M,0) \le 0$ for all $(x,t) \in \overline{Q}_\omega$; For any $(x,t) \in \overline{Q}_\omega$, and $u_2 > 0$, $F_2^0(x,t,0,u_2) < F_2^0(x,t,0,0)$.

We then have the following result on the uniform persistence of (4.5).

Theorem 4.2 Let (M1)-(M4) hold. Assume that

$$\mu^{(1)}(F_1^0(x,t,0,0)) < 0, \quad \mu^{(2)}(F_2^0(x,t,0,0)) \geq 0,$$

$$\mu^{(2)}(F_2^0(x,t,u_1^*(t,x),0)) < 0,$$

where $u_1^*(t,x)$ is the unique positive ω-periodic solution of periodic parabolic equation

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + A_1(t)u_1 = u_1 F_1^0(x,t,u_1,0) & \text{in } \ \Omega \times (0,\infty) \\ B_1 u_1 = 0 & \text{on } \ \partial\Omega \times (0,\infty) \end{cases} \tag{4.7}$$

Then system (4.5) is uniformly persistent. More precisely, there exists a $\beta > 0$ such that for any $u = (u_1, u_2) \in P_1 \times P_2$ with $u_1(x) \not\equiv 0$ and $u_2(x) \not\equiv 0$, there exists $t_0 = t_0(u) > 0$ such that $\phi(t,0,u) = (\phi_1(t,0,u), \phi_2(t,0,u))$ satisfies

$$\phi_i(t,0,u)(x) \geq \beta e_i(x) \qquad \text{for } t \geq t_0, \ x \in \overline{\Omega} \text{ and } i = 1,2.$$

where

$$e_i(x) = \begin{cases} e(x) & \text{if } B_i v = v \\ 1 & \text{if } B_i v = \dfrac{\partial v}{\partial n} + b_0^{(i)} v \end{cases}$$

$e \in C^2(\overline{\Omega})$ is given such that for $x \in \Omega$, $e(x) > 0$ and for $x \in \partial\Omega$, $e(x) = 0$ and $\dfrac{\partial e}{\partial n} < -\gamma < 0$.

Proof. By (M2) and (M3), it easily follows that $\phi(t,s,u)$ and $\phi_0(t,s,u)$ exists globally on $[s,\infty)$ and are uniformly bounded in $X = P_1 \times P_2$. Therefore, by Proposition 3.1, $\phi(t,s,u)$, $t \geq 0$, is asymptotic to ω-periodic semiflow $T(t) = \phi_0(t,0,\cdot), t \geq 0$, and for any $u \in X$, $\gamma^+(u) = \{T_n(u); n \geq 0\}$, where $T_n(u) = \phi(n\omega, 0, u), n \geq 0$, is precompact in X, and hence its ω-limit set $\omega(u)$ exists. Let $S = T(\omega) : X \to X$, then $T_n : X \to X, n \geq 0$, is an asymptotically autonomous discrete dynamical process with the limit autonomous discrete semiflow $S^n : X \to X, n \geq 0$.

Let $X_0 = \{(u_1,u_2) \in X; \quad u_i(x) \not\equiv 0, \quad i = 1,2\}$ and $\partial X_0 = \{(u_1,u_2) \in X; u_1(x) \equiv 0 \text{ or } u_2(x) \equiv 0\}$, then $X = X_0 \cup \partial X_0$, X_0 and ∂X_0 are relatively open and closed in X, respectively. By the proof of [23, Theorem 3.1], $S : X \to X$ is point dissipative, compact and uniformly persistent with respect to $(X_0, \partial X_0)$, and then, by [23, Theorem 2.1], S admits a global attractor A_0 in X_0, which is globally asymptotically stable in X_0. Let $M_1 = (0,0)$, $M_2 = (u_1^*(0), 0)$ and A_∂ be the maximal compact invariant set of S in ∂X_0, then $\tilde{A}_\partial = \bigcup_{x \in A_\partial} \omega_S(x) = \{M_1, M_2\}$, and, by the proof of [23,Theorem 3.1], $M_1 \cup M_2$ is an isolated and acyclic covering of $\tilde{A}_\partial$ in ∂X_0. Moreover, by Proposition 4.1, $\widetilde{W}^s(M_i) \cap X_0 = \emptyset$, $i = 1, 2$. By Theorem 2.5, it follows that $\omega(u) \subseteq A_0$ for any $u \in X_0$. Then, by Theorem 3.1,

$$\lim_{t \to \infty} d(\phi(t, 0, u), T(t)A_0) = 0.$$

In particular, since $T(\omega)A_0 = A_0$ and $T(t)$ is an ω- periodic semiflow, $\lim_{t\to\infty} d(\phi(t, 0, u), A_0^*) = 0$, where $A_0^* = \bigcup_{t \in [0,\omega]} T(t)A_0 = T([0, \omega] \times A_0)$.

Since $A_0 = T(\omega)A_0, A_0^* = T((0, \omega] \times A_0)$, and hence, for any $u \in A_0^*$, there exist $v \in A_0 \subseteq A_0^*$ and $t \in (0, \omega]$ such that $u = T(t)v = T(t, v)$. By using the compactness of A_0^* and the fact that $E_1 \times E_2 = X_\beta^{(1)} \times X_\beta^{(2)} \hookrightarrow C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$, and by a similar argument to [2, Lemma 3.6, Corollary 3.7 and Remark 3.8] and [23,Theorem 3.1], we can prove the required uniform persistence of system (4.5) in the theorem.

This completes the proof.

Remark 4.2 By the same argument as in Theorem 4.2, we can prove the uniform persistence of (4.5) in the case that the predator u_2 may have not self-limitation(see [23, Theorem 3.1] for some related details in periodic case). Moreover, a similar approach to that of Theorem 4.2 can be used to discuss the uniform persistence of asymptotically periodic two species Kolmogorov competition parabolic systems.

We have discussed the global asymptotic behavior of some asymptotically periodic parabolic systems in this section. Clearly, a similar approach can also be used to discuss some asymptotically periodic systems of ordinary differential equations when the asymptotic behavior of their limiting periodic systems is well understood(e.g., periodic Kolmogrov and Lotka-Voltera systems). In particular, due to certain conservation principle(see,e.g., [17] and related references therein), the periodically operated chemostat and gradostat models can be studied under the setting of asymptotically periodic systems.

References

[1] N. D. Alikakos, An application of the invariant principle to reaction-diffusion equations, J. Differential Equations, 33(1979), 201-225.

[2] R. S. Cantrell, C. Cosner and V. Hutson, Permanence in ecological sysetms with spatial heterogeneity, Proc. Roy. Soc. Edinburgh Sect. A, 123 (1993), 533-559.

[3] C. Castillo-Chavez and H. R. Thieme, Asymptotically autonomous epidemic models, Mathematical Population Dynamics: Analysis of Heterogeneity, Volume one: Theory of Epidemics (O. Arino, D. Axelrod, M. Kimmel and M. Langlais, eds.), Wuerz, 1995, 33-49.

[4] E. N. Dancer and P. Hess, Stability of fixed points for order-preserving discrete-time dynamical systems, J. reine angew. Math., 419(1991), 125-139.

[5] H. I. Freedman and J. W. -H. So, Persistence in discrete semi-dynamical systems, SIAM J. Math. Anal., 20(1989), 930-938.

[6] T. Fujimoto and U. Krause, Asymptotic properties for inhomogeneous iterations of nonlinear operators, SIAM J. Math. Anal., 19(1988), 841-853.

[7] T. Fujimoto and U. Krause, Stable inhomogeneous iterations of nonlinear positive operators on Banach spaces, SIAM J. Math. Anal., 25(1994), 1195-1202.

[8] J. K. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys and Monographs 25, Amer. Math. Soc., Providence, RI, 1988.

[9] J. K. Hale and O. Lopes, Fixed point theorems and dissipative processes, J. Differential Equations, 13(1973), 391-402.

[10] J. K. Hale and P. Waltman, Persistence in infinite-dimensional systems, SIAM J. Math. Anal., 20(1989), 388-395.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. 840, Springer-Verlag, Berlin, 1981.

[12] P. Hess, Periodic-parabolic Boundary Value Problems and Positivity, Pitman Research Notes in Math., Series 247, Longman Scientific and Technical, 1991.

[13] P. Hess, On the asymptotically periodic Fisher equation, in Progress in Partial Differential Equations: elliptic and parabolic problems, Pitman Res. Notes Math., Ser. 266, Longman Sci. Tech., Harlow, 1992, 24-33.

[14] J. Hofbauer and J. W. -H. So, Uniform persistence and repellors for maps, Proc. Amer. Math. Soc., 107(1989), 1137-1142.

[15] J. P. Lasalle, The Stability of Dynamical Systems, Society for Industrial and Applied Math., Philadelphia, 1976.

[16] K. Mischaikow, H. L. Smith and H. R. Thieme, Asymptotically autonomous semiflows: chain recurrence and Liapunov functions, Transaction of AMS, 347(1995), 1669-1685.

[17] H. L. Smith and P. Waltman, The Theory of the Chemostat, Cambridge University Press, 1995.

[18] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer, Berlin, 1983.

[19] H. R. Thieme, Convergence results and Poincaré-Bendixson trichotomy for asymptotically autonomous differential equations, J. Math. Biology, 30(1992), 755-763.

[20] H. R. Thieme, Aymptotically autonomous differential equations in the plane, Rocky Mountain J. Math., 24(1994),351-380.

[21] H. R. Thieme, Aymptotically autonomous differential equations in the plane (II): strict Poncaré-Bendixson type results, Differential and Integral Equations, 7(1994), 1625-1640.

[22] X. -Q. Zhao, Global attractivity and stability in some monotone discrete dynamical systems, Bull. Austral. Math. Soc., 53(1996), 305-324.

[23] X.-Q. Zhao, Uniform persistence and periodic coexistence states in infinite-dimensional periodic semiflows with applications, Canadian Applied Math. Quarterly, 3(1995), 473-495.