Notes on integral equations with time delay

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For simplicity, we consider the time-delayed integral equation

$$\begin{cases} u(t) = \int_{t-\tau}^{t} a(s)u(s)ds, & t \geq 0, \\ u_0 = \varphi \in C([-\tau, 0], \mathbb{R}) := X. \end{cases} \tag{0.1}$$

Here we assume that $a(t) \geq 0, \forall t \in \mathbb{R}$.

Lemma 0.1. *System* (0.1) *has a (unique) solution if and only if* φ *satisfies*

$$\varphi(0) = \int_{-\tau}^{0} a(s)\varphi(s)ds. \tag{0.2}$$

Proof. (a) If (0.1) has a solution $u(t)$, then letting $t \to 0^+$ in the first equation, we obtain $u(0) = \int_{-\tau}^{0} a(s)u(s)ds$, which implies (0.2) because $u_0 = \varphi$.

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(b) Assume that $\varphi \in X$ satisfies (0.2). Consider the linear differential equation with delay

$$\begin{cases} u'(t) = a(t)u(t) - a(t-\tau)u(t-\tau), \quad t > 0, \\ u_0 = \varphi. \end{cases} \tag{0.3}$$

Let $u(t)$ be the unique solution of (0.3). Then we have

$$\frac{d}{dt}\left(u(t) - \int_{t-\tau}^{t} a(s)u(s)ds\right) = 0, \quad \forall\ t \geq 0.$$

Thus, $u(t) = \int_{t-\tau}^{t} a(s)u(s)ds + C, \forall\ t \geq 0$. Letting $t = 0$, we obtain

$$u(0) = \int_{-\tau}^{0} a(s)\varphi(s)ds + C,$$

and hence, $C = \varphi(0) - \int_{-\tau}^{0} a(s)\varphi(s)ds = 0$. It follows that

$$u(t) = \int_{t-\tau}^{t} a(s)u(s)ds, \quad \forall\ t \geq 0,$$

that is, $u(t)$ satisfies equation (0.1). $\square$

Remark 0.1. *From Lemma 0.1, we see that if initial function φ does not satisfies* (0.2)*, then the integral equation* (0.1) *has no solution.*

Lemma 0.2. (The comparison principle) *If $\varphi_1 \geq \varphi_2$ and φ_i satisfies* (0.2)*, $i = 1, 2$, then $u(t, \varphi_1) \geq u(t, \varphi_2), \forall\ t \geq 0$. Here $u(t, \varphi_i)$ is the unique solution of integral equation* (0.1) *with $\varphi = \varphi_i$, $i = 1, 2$.*

Lemma 0.3. (The comparison principle) *If a continuous function $u(t)$ satisfies*

$$u(t) \geq \int_{t-\tau}^{t} a(s)u(s)ds, \quad \forall\ t \geq 0,$$

and there exists $\varphi \in X$ such that $u(s) \geq \varphi(s), \forall\ s \in [-\tau, 0]$, and φ satisfies (0.2)*, then we have*

$$u(t) \geq u(t, \varphi), \quad \forall\ t \geq 0.$$

Here $u(t, \varphi)$ is the unique solution of integral equation (0.1)*.*

Lemma 0.4. (The comparison principle) *If a continuous function $v(t)$ satisfies*

$$v(t) \le \int_{t-\tau}^{t} a(s)u(s)ds, \quad \forall\ t \ge 0,$$

and there exists $\varphi \in X$ such that $v(s) \le \varphi(s), \forall\ s \in [-\tau, 0]$, and φ satisfies (0.2)*, then we have*

$$v(t) \le u(t, \varphi), \quad \forall\ t \ge 0.$$

Here $u(t, \varphi)$ is the unique solution of integral equation (0.1)*.*

Remark 0.2. *Whenever we mention solutions of integral equation* (0.1)*, we must verify that the initial function satisfies the so-called matching condition* (0.2)*, see Remark 0.1.*

As an example, we consider the integral equation

$$\begin{cases} u(t) = \int_{t-\tau}^{t} u(s)ds, \\ u_0 = \varphi \in X. \end{cases} \tag{0.4}$$

Clearly, $u(t) \equiv 1$ is a solution of the delay differential equation

$$\begin{cases} u'(t) = u(t) - u(t-\tau), \\ u_0 = 1. \end{cases} \tag{0.5}$$

However, $u(t) \equiv 1$ is not a solution of (0.4) if $\tau \ne 1$. Note that for (0.4), the matching condition (0.2) reduces to $\varphi(0) = \int_{-\tau}^{0} \varphi(s)ds$.

Remark 0.3. *The phase space for integral equation* (0.1) *is*

$$Y = \left\{ \varphi \in X : \varphi(0) = \int_{-\tau}^{0} a(s)\varphi(s)ds \right\}.$$

By Lemma 0.2, it follows that the integral equation (0.1) *admits the comparison principle on Y. Thus, the delay differential equation* (0.3) *admits the comparison principle on Y rather than X.*