Notes on Wang and Zhao's JDDE paper

Discontinuous linear systems:

The theory of R_0 developed in Wang and Zhao's paper [2] also applies to the case where the periodic coefficients in a linear system have finite many discontinuous points.

Indeed, the theory of evolution matrix, the constant variation formula, and the Floquet theory for the linear T-periodic system $x' = A(t)x$ are valid provided that $A(t)$ is T-periodic and Lebesgue integrable. Here the equation is required to be satisfied almost for all $t \in [0, T]$. For the Caratheodory conditions on the existence, uniqueness, and continuation of mild solutions, we refer to Hale's book [1]. Clearly, a periodic function with finite many discontinuous points is Lebesgue integrable.

Due to the same reason, [3, Lemma 2.1] is also applicable if periodic coefficients have finite many discontinuous points.

A numerical algorithm to compute R_0:

Consider the linear ω-periodic ODE system

$$\frac{dw}{dt} = \left(-V(t) + \frac{1}{\lambda}F(t)\right) w, \quad w \in \mathbb{R}^m \tag{1}$$

with parameter $\lambda \in (0, \infty)$. Let $W(t, \lambda), t \geq 0$, be the standard fundamental matrix of (1) with $W(0, \lambda) = I$.

By [2, Theorem 2.1 (ii)], we know that if $R_0 > 0$, then $\lambda = R_0$ is the unique solution of $\rho(W(\omega, \lambda)) = 1$.

For any specific value of λ, one can numerically compute all eigenvalues of the matrix $W(\omega, \lambda)$, and hence, the spectral radius, $\rho(W(\omega, \lambda))$, of $W(\omega, \lambda)$. I believe that there exists such a software. Recall how people study the stability of a periodic orbit numerically: first linearize the given autonomous ODE system at an ω-periodic solution to obtain a linear periodic ODE system, and then one needs to compute the Floquet multipliers, that is, all eigenvalues of the matrix $W(\omega, \lambda)$ associated with the resultant linear system.

Let $f(\lambda) := \rho(W(\omega, \lambda))$. Since $F(t)$ is nonnegative and $-V(t)$ is cooperative, it follows that $f(\lambda)$ is continuous and non-increasing in $\lambda \in (0, \infty)$. Further, $\lim_{\lambda \to \infty} f(\lambda) = \rho(\Phi_{-V}(\omega)) < 1$.

(1) Choose two positive numbers $a_0 < b_0$ such that $f(a_0) > 1 > f(b_0)$. If there is no such a_0, then [2, Theorem 2.1 (iii)] implies that $R_0 = 0$.

(2) Define two sequences a_n and b_n by induction: If $f(\frac{1}{2}(a_n + b_n)) \geq 1$, define $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = b_n$; Otherwise, define $a_{n+1} = a_n$ and $b_{n+1} = \frac{1}{2}(a_n + b_n)$. It follows that $a_n \leq b_n$, $a_{n+1} \geq a_n$, $b_{n+1} \leq b_n$, and $f(a_n) \geq 1 \geq f(b_n)$ for all n.

(3) Note that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and $b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$. Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lambda_0 > 0$. Since $f(a_n) \geq 1 \geq f(b_n)$ for all n, we have $f(\lambda_0) \geq 1 \geq f(\lambda_0)$, and hence, $f(\lambda_0) = 1$. Consequently, we have $R_0 = \lambda_0$.

(4) Since $a_n \leq R_0 \leq b_n$, we see that $|a_n - R_0| \leq b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$, and $|b_n - R_0| \leq b_n - a_n = \frac{1}{2^n}(b_0 - a_0)$. Given an error tolerance ϵ, we can choose an $N > 0$ such that $\frac{1}{2^N}(b_0 - a_0) \leq \epsilon$. Thus, we have $R_0 \approx a_N$ or $R_0 \approx b_N$.

References

[1] J. K. Hale, *Ordinary Differential Equations*, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[2] W. Wang and X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, *J. Dyn. Diff. Eqns.*, **20**(2008), 699–717.

[3] F. Zhang and X.-Q. Zhao, A periodic epidemic model in a patchy environment, *J. Math. Anal. Appl.*, **325**(2007), 496-516.