Lecture Notes on Abstract Persistence

(Xiaoqiang Zhao, Math 6104, MUN)

Let (X, d) be a complete metric space with metric d, and X_0 be an open subset of X. Set $\partial X_0 := X \setminus X_0$. Clearly, ∂X_0 is a closed subset of X, $X = X_0 \cup \partial X_0$, and $X_0 \cap \partial X_0 = \emptyset$.

Definition 1. Let $\{\Phi(t)\}_{t\geq 0}$ be a semiflow on X with $\Phi(t)X_0 \subset X_0$ for all $t \geq 0$. $\Phi(t)$ is said to be uniformly persistent with respect to X_0 if there exists $\eta > 0$ such that $\liminf_{t\to\infty} d(\Phi(t)x, \partial X_0) \geq \eta$ for all $x \in X_0$.

Note that in [1], it is also assumed that ∂X_0 is positively invariant for $\Phi(t)$, that is, $\Phi(t)\partial X_0 \subset \partial X_0$ for all $t \geq 0$. However, this assumption is not satisfied for some population models. Thus, we will use M_∂ to denote the maximal positively invariant set of the semiflow $\Phi(t)$ in ∂X_0. It is easy to see that

$$M_\partial = \{x \in \partial X_0 : \ \Phi(t)x \in \partial X_0, \ \forall t \geq 0\}.$$

Example 1. Let $\Phi(t)$ be the solution semiflow of a scalar FDE model, that is, $\Phi(t)\phi = u_t(\phi)$, $\forall \phi \in X := C([-\tau, 0], \mathbb{R}_+)$. Let $X_0 = X \setminus \{0\}$ and $\partial X_0 = \{0\}$. Assume that $\Phi(t)$ is uniformly persistent with respect to X_0 in the sense of Definition 1. Then we have

$$\begin{aligned}
\eta &\leq \liminf_{t\to\infty} d(\Phi(t)\phi, \partial X_0) \\
&= \liminf_{t\to\infty} \max_{\theta\in[-\tau,0]} u_t(\phi)(\theta) \\
&= \liminf_{t\to\infty} \max_{\theta\in[-\tau,0]} u(t+\theta, \phi) \\
&= \liminf_{t\to\infty} \max_{s\in[t-\tau,t]} u(s, \phi), \ \forall \phi \in X_0,
\end{aligned}$$

which is different from our desired practical persistence in the sense that there exists $\bar{\eta} > 0$ such that $\liminf_{t\to\infty} u(t, \phi) \geq \bar{\eta}$, $\forall \phi \in X_0$. Note that the acyclicity theorems in [1] are for abstract persistence. It remains a problem how to obtain the practical persistence for a FDE model from the abstract persistence of its solution semiflow.

Example 2. Let $\Phi(t)$ be the solution semiflow of a scalar reaction-diffusion model subject to the Robin type boundary condition, that is, $\Phi(t)\phi = u(t, \cdot, \phi)$, $\forall \phi \in X := C(\overline{\Omega}, \mathbb{R}_+)$. Let $X_0 = X \setminus \{0\}$ and $\partial X_0 = \{0\}$. Assume that $\Phi(t)$ is uniformly persistent with respect to X_0 in the sense of Definition 1. Then we have

$$\begin{aligned}
\eta &\leq \liminf_{t\to\infty} d(\Phi(t)\phi, \partial X_0) \\
&= \liminf_{t\to\infty} \max_{x\in\overline{\Omega}} u(t, x, \phi), \ \forall \phi \in X_0,
\end{aligned}$$

which is different from our desired practical persistence in the sense that there exists $\bar{\eta} > 0$ such that $\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u(t,x) \ge \bar{\eta}$, $\forall \phi \in X_0$. Again we have a problem on how to obtain the practical persistence for a PDE model from the abstract persistence of its solution semiflow.

To solve the above mentioned problem, one can directly use the following persistence theorem, which is from [2, Theorem 3].

Definition 2. A lower semicontinuous function $p : X \to \mathbb{R}_+$ is called a generalized distance function for the semiflow $\Phi(t) : X \to X$ if for every $x \in (X_0 \cap p^{-1}(0)) \cup p^{-1}(0,\infty)$, we have $p(\Phi(t)x) > 0$, $\forall t > 0$.

Theorem 1. Let p be a generalized distance function for the semiflow $\Phi(t) : X \to X$ with $\Phi(t)X_0 \subset X_0$ for all $t \ge 0$. Assume that

(P1) $\Phi(t) : X \to X$ has a global attractor A;

(P2) There exists a finite sequence $M = \{M_1, \ldots, M_k\}$ of disjoint, compact, and isolated invariant sets in ∂X_0 with the following properties:

- (a) $\cup_{x\in M_\partial}\omega(x) \subset \cup_{i=1}^k M_i$;
- (b) No subset of M forms a cycle in ∂X_0;
- (c) Each M_i is isolated in X;
- (d) $W^s(M_i) \cap p^{-1}(0,\infty) = \emptyset$ for each $1 \le i \le k$.

Then there exists $\delta > 0$ such that $\liminf_{t\to\infty} p(\Phi(t)x) > \delta$ for all $x \in X_0$.

Note that the conclusion in [2, Theorem 3] is much stronger: there exists $\delta > 0$ such that for any compact chain transitive set L with $L \not\subset M_i$ for all $1 \le i \le k$, we have $\min_{x\in L} p(x) > \delta$. Indeed, for any $x_0 \in X_0$, $\omega(x_0)$ is a compact chain transitive set. By Definition 2 and the property (d), it is easy to see that $\omega(x_0) \not\subset M_i$ for all $1 \le i \le k$. Thus, we have $\min_{x\in\omega(x_0)} p(x) > \delta$, which implies the conclusion in Theorem 1.

In the case where $p(x) = d(x, \partial X_0)$, we have $p^{-1}(0) = \partial X_0$ and $p^{-1}(0,\infty) = X_0$, and hence, the p-persistence in Theorem 1 is exactly the abstract persistence in the sense of Definition 1. Thus, Theorem 1 is a generalization of [1, Theorems 4.1 and 4.2].

Answer 1. To get the practical persistence for a FDE system with initial data ϕ, we can define

$$p(\phi) := \min_{1\le i\le m} \{\phi_i(0)\}, \quad \forall \phi = (\phi_1, \ldots, \phi_m) \in X := C([-\tau, 0], \mathbb{R}^m_+).$$

Clearly, $p : X \to \mathbb{R}_+$ is continuous, and $u_t(\phi)(0) = u(t, \phi)$.

Answer 2. Let $e \in Int(C_0^1(\overline{\Omega}, \mathbb{R}_+)$ be given. To get the practical persistence for a PDE system with initial data ϕ, we can define

$$p(\phi) := \min_{1 \le i \le m} \left\{ \min_{x \in \overline{\Omega}} \phi_i(x) \right\}, \quad \forall \phi = (\phi_1, \ldots, \phi_m) \in X := C(\overline{\Omega}, \mathbb{R}_+^m),$$

in the case of Robin type boundary condition, and

$$p(\phi) := \sup\{\beta \in \mathbb{R}_+ : \ \phi_i(x) \ge \beta e(x), \, \forall x \in \overline{\Omega}, 1 \le i \le m\}, \\ \forall \phi = (\phi_1, \ldots, \phi_m) \in X := C_0(\overline{\Omega}, \mathbb{R}_+^m)$$

in the case of Dirichlet boundary condition, respectively. It is easy to see that the first p-function is continuous, and the second one is lower semicontinuous.

References

[1] J. K. Hale and P. Waltman, Persistence in infinite-dimensional systems, *SIAM J. Math. Anal.*, 20 (1989), 388–395.

[2] H. L. Smith and X.-Q. Zhao, Robust persistence for semidynamical systems, *Nonlinear Analysis, TMA*, 47(2001), 6169–6179.