Notes on periodic linear FDEs

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Lemma A. ([2, Lemma 2.6]) Assume that (E, E^+) is an ordered Banach space with E^+ being normal and $Int(E^+) \neq \emptyset$. Let $\mathcal{L}$ be a positive and bounded linear operator. If λ is an eigenvalue of $\mathcal{L}$ with a strongly positive eigenvector, then $\lambda = r(\mathcal{L})$.

Lemma B. ([1, Lemma 3.1]) Let $\mathcal{L}$ be a positive and bounded linear operator on an ordered Banach space (E, E^+) with $Int(E^+) \neq \emptyset$. If there is a positive integer n_0 such that $\mathcal{L}^{n_0}$ is compact and strongly positive on E, then $r(\mathcal{L})$ is a simple eigenvalue of $\mathcal{L}$ having a strongly positive eigenvector.

Assume that for each $1 \leq i \leq n$, (X_i, X_i^+) is an ordered Banach space with X_i^+ being normal and $Int(X_i^+) \neq \emptyset$. Let $X = \prod_{i=1}^n X_i$ and $X^+ = \prod_{i=1}^n X_i^+$. Then (X, X^+) is an ordered Banach space with X^+ being normal and $Int(X^+) \neq \emptyset$. Let $\tau \geq 0$ be a given real number. We consider an abstract linear periodic FDE on the Banach space $C := C[-\tau, 0], X)$:

$$\frac{du(t)}{dt} = L(t)u_t, \quad t \geq 0, \tag{0.1}$$

where $F(t) : C \to X$ is a linear operator for each $t \in \mathbb{R}$, and $L(t)$ is ω-periodic in $t \in \mathbb{R}$ for some $\omega > 0$. Assume that for any $\phi \in C$, equation (0.1) has a unique solution $u(t, \phi)$ satisfying $u_0 = \phi$. Define $P(t)\phi = u_t(\phi)$. It then follows that $P(t)$ is an ω-periodic semilow on C. Let $r(P(\omega))$ be the spectral radius of the Poincaré map $P(\omega)$.

Lemma 1. Assume that (0.1) admits the comparison principle, that is, $P(t)$ is a monotone periodic semiflow on C . If (0.1) has a solution $u^*(t) = e^{\mu t}v^*(t)$ such that $v^*(t+\omega) = v^*(t) \gg 0$ in X for all $t \in \mathbb{R}$. Then $r(P(\omega)) = e^{\mu\omega}$.

Proof. Define $\phi^* \in C$ by $\phi^*(\theta) = e^{\mu\theta}v^*(\theta),\ \forall\theta \in [-\tau, 0]$. Then $\phi^* \gg 0$ in C, and $u(t, \phi^*) = u^*(t), \forall t \geq -\tau$. It follows that

$$[P(\omega)\phi^*](\theta) = u(\omega + \theta, \phi^*) = e^{\mu(\omega+\theta)}v^*(\omega + \theta) = e^{\mu\omega}\phi^*(\theta),\ \forall\theta \in [-\tau, 0],$$

and hence, $P(\omega)\phi^* = e^{\mu\omega}\phi^*$. Now Lemma A implies that $r(P(\omega)) = e^{\mu\omega}$.

Let $\tau_i \in [0, \tau]$, $1 \leq i \leq n$, be given real numbers. We set

$$Y = \prod_{i=1}^{n} C([-\tau_i, 0], X_i), \quad Y^+ = \prod_{i=1}^{n} C([-\tau_i, 0], X_i^+).$$

Lemma 2. Let $P(t)$ be defined as in Lemma 1. Assume that (0.1) generates a monotone ω-periodic semiflow $\tilde{P}(t)$ on Y. If $r(\tilde{P}(\omega))$ is an eigenvalue of $\tilde{P}(\omega)$ having a strongly positive eigenvector in Y, then $r(\tilde{P}(\omega)) = r(P(\omega))$.

Proof. Let $\mu := \frac{\ln r(\tilde{P}(\omega))}{\omega}$. By the essentially same arguments as in [3, Proposition 2.1], it then follows that (0.1) has a solution $u^*(t) = e^{\mu t}v^*(t)$ such that $v^*(t + \omega) = v^*(t) \gg 0$ in X for all $t \in \mathbb{R}$. Thus, Lemma 1 implies that $r(P(\omega)) = e^{\mu\omega} = r(\tilde{P}(\omega))$.

Remark 1. If $\tilde{P}(t)$ is eventually compact and strongly monotone on Y, then the conclusion of Lemma 2 holds true. This is because Lemma B, together with the fact that $(\tilde{P}(\omega))^n = \tilde{P}(n\omega),\ \forall n \geq 0$, implies that $r(\tilde{P}(\omega))$ is a simple eigenvalue of $\tilde{P}(\omega)$ having a strongly positive eigenvector in Y.

Remark 2. In the study of nonlinear evolution systems with time-periodic delay, it is **not** necessary to choose the product space as its phase space. In some published papers, such a product space was used to find the positive solution $u^*(t) = e^{\mu t}v^*(t)$ for the linear system associated with the definition of the basic reproduction ratio R_0, which was introduced in [4, 2].

References

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