Lecture Notes on Limiting Systems and Chain Transitive Sets

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1 Motivations

Example 1.1. Let D be a closed subset of $\mathbb{R}^n$. We consider the nonautonomous ordinary differential system:

$$\begin{aligned} &\frac{dx}{dt} = f(t,x), \ t \geq 0, \\ &x(0) = x_0 \in D. \end{aligned} \tag{1.1}$$

Assume that $\lim_{t\to\infty} f(t,x) = f_0(x)$ uniformly for x in any bounded subset of D. Then we have a limiting autonomous system:

$$\begin{aligned} &\frac{dx}{dt} = f_0(x), \ t \geq 0, \\ &x(0) = x_0 \in D. \end{aligned} \tag{1.2}$$

Problem. Under what conditions can we lift the long-time properties of solutions of the limiting system (1.2) to the nonautonomous system (1.1)?

To solve the above problem, one may use the theory of asymptotically autonomous semiflows, see [3, 2] and the references therein.

In applications, we may also meet asymptotically autonomous FDEs and PDEs.

In this theory, the domain of the asymptotically autonomous system is assumed to be the same as that of the limiting autonomous system.

Example 1.2. Consider the single species growth model in a chemostat:

$$\begin{aligned} \frac{dS}{dt} &= D(S^0 - S) - xP(S) \\ \frac{dx}{dt} &= xP(S) - Dx \\ (S(0), x(0)) &= (S_0, x_0) \in \mathbb{R}^2_+. \end{aligned} \tag{1.3}$$

Here D is the dilution (or washout) rate, S^0 is the inout nutrient concentration, $P(S)$ is the per capita nutrient uptake function. In particular, we take $P(S) = \frac{mS}{a+S}$, where m is maximal growth rate, and a is the Michaelis-Menten (or half-saturation) constant. Both a and m can be measured experimentally.

It is easy to see that $\mathbb{R}^2_+$ is positively invariant for system (1.3).

Let $\Sigma = S + x$. Then system (1.3) is equivalent to the following one:

$$\begin{aligned} \frac{d\Sigma}{dt} &= DS^0 - D\Sigma \\ \frac{dx}{dt} &= xP(\Sigma - x) - Dx \\ (\Sigma(0), x(0)) &= (\Sigma_0, x_0) \in \Omega := \{(\Sigma, x) : \Sigma \geq x \geq 0\}. \end{aligned} \tag{1.4}$$

It then follows that Ω is positively invariant for system (1.4). Clearly, $x(t)$ satisfies the following nonautonomous equation:

$$\frac{dx}{dt} = xP(\Sigma(t) - x) - Dx. \tag{1.5}$$

Let $\Omega(t) := [0, \Sigma(t)]$, $\forall t \geq 0$. It is easy to see that for any initial value $x(0) \in \Omega(0)$, system (1.5) has a unique solution $x(t)$ such that $x(t) \in \Omega(t)$, $\forall t \geq 0$.

Since $\lim_{t \to \infty} \Sigma(t) = S^0$, we have the following limiting system:

$$\frac{dx}{dt} = xP(S^0 - x) - Dx. \tag{1.6}$$

Let $\omega = \omega(\Sigma_0, x_0)$ be the omega limit set of (Σ_0, x_0) for the solution semiflow Φ_t of (1.4). Since $(\Sigma(t), x(t)) \in \Omega$, $\forall t \geq 0$, we have $\Sigma(t) \geq x(t) \geq 0$, and hence, $\omega = \{S^0\} \times \tilde{\omega}$ with $\tilde{\omega} \subset [0, S^0]$. Let Q_t be the solution semiflow of (1.6) on $[0, S^0]$. Since $\Phi_t(\omega) = \omega$, $\forall t \geq 0$, we have

$$\Phi_t(S^0, \bar{x}) = (S^0, Q_t(\bar{x})), \ \forall (S^0, \bar{x}) \in \omega, \, t \geq 0.$$

It then follows that $Q_t(\tilde{\omega}) = \tilde{\omega}$, $\forall t \geq 0$.

Note that the nonautonomous system (1.5) has a time-dependent domain $\Omega(t)$, while the limiting system (1.6) has the domain $[0, S^0]$. Thus, we cannot directly use the theory of asymptotically autonomous systems.

2 Chain transitive sets

Let (X, d) be a complete metric space with metric d, and $\Phi(t) : X \to X$, $t \geq 0$, be a continuous-time semiflow.

Definition 2.1. *A nonempty invariant set $A \subset X$ for $\Phi(t)$ (i.e., $\Phi(t)A = A$, $\forall t \geq 0$) is said to be internally chain transitive if for any $a, b \in A$ and any $\epsilon > 0, t_0 > 0$, there is a finite sequence*

$$\{x_1 = a, x_2, \ldots, x_{m-1}, x_m = b; \, t_1, \ldots, t_{m-1}\}$$

with $x_i \in A$ *and* $t_i \geq t_0, 1 \leq i \leq m-1$*, such that* $d(\Phi(t_i, x_i), x_{i+1}) < \epsilon$ *for all* $1 \leq i \leq m-1$*. The sequence* $\{x_1, \ldots, x_m; t_1, \ldots, t_{m-1}\}$ *is called an* (ϵ, t_0)*-chain in* A *connecting* a *and* b*.*

Lemma 2.1([1]) *Let* $\Phi(t) : X \to X$*,* $t \geq 0$*, be an autonomous semiflow. Then the omega (alpha) limit set of any precompact positive (negative) orbit is internally chain transitive.*

Exercise 2.1 Show that the set $\tilde{\omega}$ in Example 1.2 is a chain transitive set for the solution semiflow of the limiting system (1.6) on $[0, S^0]$.

Recall that a continuous mapping $\Phi : \Delta_0 \times X \to X$, $\Delta_0 = \{(t,s) : 0 \leq s \leq t < \infty\}$, is called a nonautonomous semiflow if Φ satisfies the following properties:

(i) $\Phi(s, s, x) = x, \quad \forall s \geq 0,\ x \in X$;

(ii) $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x), \quad \forall t \geq s \geq r \geq 0$.

Definition 2.2. *A nonautonomous semiflow* $\Phi : \Delta_0 \times X \to X$ *is called asymptotically autonomous with limit semiflow* $Q(t) : X \to X, t \geq 0$*, if*

$$\Phi(t_j + s_j, s_j, x_j) \to Q(t)x, \text{ as } j \to \infty,$$

for any three sequences $t_j \to t, s_j \to \infty, x_j \to x$*, with* $x, x_j \in X$*.*

Lemma 2.2 ([2, 1])*Let* $\Phi(t,s) : X \to X$ *be an asymptotically autonomous semiflow with limit semiflow* $Q(t) : X \to X$*. Then the omega limit set of any precompact orbit of* $\Phi(t,s)$ *is internally chain transitive for* $Q(t)$*.*

Exercise 2.2 Show that the nonautonomous semiflow associated with system (1.1) is asymptotic to the autonomous semiflow associated with system (1.2).

Theorem 2.1 ([1]) *Let A be an attractor and C a compact internally chain transitive set for the autonomous semiflow $Q(t) : X \to X$. If $C \cap W^s(A) \neq \emptyset$, then $C \subset A$.*

Theorem 2.2 ([1])*Assume that each equilibrium of the autonomous semiflow $Q(t) : X \to X$ is an isolated invariant set, that there is no cyclic chain of equilibria, and that every precompact orbit converges to some equilibrium of $Q(t)$. Then any compact internally chain transitive set is an equilibrium of $Q(t)$.*

For the theory of asymptotically periodic systems, we refer to [4, 5, 6].

3 An application

It is easy to obtain the global dynamics of system (1.6) on $[0, S^0]$.

Lemma 3.1 Assume that $P'(s) > 0$, $\forall s \geq 0$. Then the following statements are valid:

(a) If $P(S^0) \leq D$, then $x = 0$ is globally asymptotically stable for system (1.6) in $[0, S^0]$.

(b) If $P(S^0) > D$, then system (1.6) has a positive equilibrium $x^* \in (0, S^0)$ and $x = x^*$ is globally asymptotically stable for system (1.6) in $(0, S^0]$.

For the model system (1.3), we have the following threshold result.

Theorem 3.1 Assume that $P'(s) > 0$, $\forall s \geq 0$, and let $(S(t), x(t))$ be the solution of system (1.3). Then the following statements are valid:

(i) If $P(S^0) \leq D$, then $\lim_{t\to\infty}(S(t), x(t)) = (S^0, 0)$ for all $S(0) \geq 0$ and $x(0) \geq 0$.

(ii) If $P(S^0) > D$, then there exists $x^* \in (0, S^0)$ with $P(S^0 - x^*) = D$ such that $\lim_{t\to\infty}(S(t), x(t)) = (S^0 - x^*, x^*)$ for all $S(0) \geq 0$ and $x(0) > 0$.

Proof. Let $\Phi(t)$ be the solution semiflow of system (1.4) on Ω, and $Q(t)$ be the solution semiflow of system (1.6) on $[0, S^0]$. Let ω and $\tilde{\omega}$ be defined as in Example 1.2. By Exercise 2.1, $\tilde{\omega}$ is an internally chain transitive set for $Q(t)$ on $[0, S^0]$.

In the case where $P(S^0) \leq D$, we see from Lemma 3.1 that $W^s(0) = [0, S^0]$, and hence, $\tilde{\omega} \cap W^s(0) \neq \emptyset$. By Theorem 2.1, it then follows that $\tilde{\omega} = \{0\}$, and hence, $\omega = (S^0, 0)$. This implies that $\lim_{t\to\infty}(\Sigma(t), x(t)) = (S^0, 0)$, and $\lim_{t\to\infty} S(t) = \lim_{t\to\infty}(\Sigma(t) - x(t)) = S^0$.

In the case where $P(S^0) > D$, we see from Lemma 3.1 that $W^s(x^*) = (0, S^0]$. Since $x(0) > 0$, we have $x(t) > 0$, $\forall t \geq 0$ (**why?**). Now we show that $\tilde{\omega} \cap W^s(x^*) \neq \emptyset$. Assume, by contradiction, that $\tilde{\omega} \cap W^s(x^*) = \emptyset$. Then $\tilde{\omega} = \{0\}$, and hence, $\omega = (S^0, 0)$. Thus, we have $\lim_{t\to\infty}(\Sigma(t), x(t)) = (S^0, 0)$.

Since $\lim_{t\to\infty}(P(\Sigma(t) - x(t)) - D) = P(S^0) - D > 0$, there exists $T > 0$

such that

$$P(\Sigma(t) - x(t)) - D > \frac{1}{2}\left(P(S^0) - D\right) > 0, \quad \forall t \geq T.$$

Then we have

$$x'(t) \geq x(t) \cdot \frac{1}{2}\left(P(S^0) - D\right), \quad \forall t \geq T.$$

This implies that $x(t) \to \infty$ as $t \to \infty$, a contradiction. Thus, $\tilde{\omega} \cap W^s(x^*) \neq \emptyset$. By Theorem 2.1, it follows that $\tilde{\omega} = x^*$, and hence, $\omega = (S^0, x^*)$. Thus, we have $\lim\limits_{t\to\infty}(\Sigma(t), x(t)) = (S^0, x^*)$, and $\lim\limits_{t\to\infty} S(t) = \lim\limits_{t\to\infty}(\Sigma(t) - x(t)) = S^0 - x^*$.

Exercise 3.1 Use Theorem 2.2 to prove the conclusion (ii) in Theorem 3.1.

References

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