Notes on Liang and Zhao's 2010 JFA paper

A remark on compactness:

All results in Liang and Zhao's paper [2] are still valid if the assumption (A3) is replaced by the following weaker one:

(A3′) For any number $s \geq 0$, there exists $k = k(s) \in [0, 1)$ such that for any interval $I = [a, b]$ of the length s and any $\mathcal{U} \subset \mathcal{K}$ with $T_y[\mathcal{U}] = \mathcal{U}, \forall y \in \mathcal{H}$, we have $\alpha((Q[\mathcal{U}])_I) \leq k(s)\alpha(\mathcal{U}_I)$. Here α denotes the Kuratowski measure of noncompactness in $\mathcal{C}_I$.

To see why this observation is true, it suffices to check the preliminary results where the assumption (A3) is used.

(1) [2, Lemma 2.4] is true since its proof only used the above (A3′). Note that $\tilde{A}$, defined in the proof, satisfies $T_y[\tilde{A}] = \tilde{A}, \forall y \in \mathcal{H}$.

(2) To prove [2, Proposition 2.4] under (A3′), we define $\hat{A}_0 = \mathcal{K}$ and

$$\hat{A}_i = \left\{ T_y[u] : \ u \in \overline{\bigcup_{n=1}^{\infty} R_{c,1/n}[\hat{A}_{i-1}]}, \ y \in \mathcal{H} \right\}, \quad i \geq 1.$$

Clearly, $T_y[\hat{A}_i] = \hat{A}_i$, $\forall y \in \mathcal{H}$, $i \geq 0$, and $A_i \subset \hat{A}_i$, $\forall i \geq 0$. Then the arguments in [2, Proposition 2.4] (after replacing A_i with $\hat{A}_i$) imply that

$$\alpha(\overline{\bigcup_{n=1}^{\infty} (R_{c,1/n}[\hat{A}_i])_I}) \leq k^{i+1} m.$$

Since

$$(A_{i+1})_I \subset \overline{\bigcup_{n=1}^{\infty} (R_{c,1/n}[A_i])_I} \subset \overline{\bigcup_{n=1}^{\infty} (R_{c,1/n}[\hat{A}_i])_I},$$

it follows that $\alpha((A_{i+1})_I) \leq k^{i+1} m$.

(3) [2, Proposition 2.5] is true because its proof only needs [2, Lemma 2.3] and the new Proposition 2.4.

An α-contraction result:

The construction of the equivalent norm in [1, Theorem 4.1.1] is not right. There are two reasons. First, $S(t)$ is not a semigroup on C, e.g., $S(0) \neq I$. Second, $|\cdot|^*$ is not a norm in C since $|c|^* = 0$ for any constant function $c \in C$. However, the conclusion of [1, Theorem 4.1.1] is still valid, that is, there does exist an equivalent norm in C such that $S(t)$ is an α-contraction for each $t > 0$. The third paragraph on page 879 in paper [2] provides a modified proof with an obvious change of notations.

Some typos:

(1) Page 864, line -3, the left $(A_i)_I$ should be $(A_{i+1})_I$.

(2) Page 881, in the paragraph below Theorem 5.2, $\lim_{x\to-\infty} |V(x,x) - \beta(x)| = 0$ and $\lim_{x\to+\infty} |V(x,x)| = 0$ should be $\lim_{\xi\to-\infty} |V(\xi,x) - \beta(x)| = 0$ and $\lim_{\xi\to+\infty} |V(\xi,x)| = 0$, respectively; $\lim_{x\to+\infty} |V(x,x) - \beta(x)| = 0$ and $\lim_{x\to-\infty} |V(x,x)| = 0$ should be $\lim_{\xi\to+\infty} |V(\xi,x) - \beta(x)| = 0$ and $\lim_{\xi\to-\infty} |V(\xi,x)| = 0$, respectively.

References

[1] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[2] X. Liang and X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, *Journal of Functional Analysis*, **259**(2010), 857–903.