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A NOTE ON MINORS OF A GENERALIZED HANKEL MATRIX

Hamid Usefi

Rashid Zaare-Nahandi

Institute for Advanced Studies in Basic Sciences
P.O.Box 45195-159, Zanjan, Iran

Abstract

A generalization of the well known Hankel matrix, which is called d-Hankel matrix is introduced by Simis and Machado [4]. We show that any t-minor of the d-Hankel matrix can be written as a linear combination of maximal minors of another matrix which is again a d-Hankel matrix.

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1 Introduction

Hankel matrices have several applications in various fields ranging from engineering to computer science. Ideal of 2-minors (respectively, t-minors, for $t > 2$) of a generic Hankel matrix is defining ideal of rational normal curve (respectively, secant varieties of rational normal curves) in a projective space [5], and it is widely studied in algebraic geometry. Ideal of 2-minors of concatenation of d truncated $2 \times n_i$, $(i = 1, \ldots, d)$, Hankel matrices is defining ideal of rational normal scroll in a projective space [3]. After a suitable reordering the columns of this matrix we obtain a d-Hankel matrix. In some topics of mathematics, specially theory of determinantal rings, study of maximal minors of a matrix is more simple than study of t-minors of the matrix for arbitrary t (see [1]). These are our motivation to study the generalization of Hankel matrices and to find a converted form for them where any t-minor of the generalized Hankel matrix is a linear combination of maximal minors of the converted matrix.

In the following, we define a generic Hankel matrix and a generic d-Hankel matrix. It is clear that, all results are true if we consider a Hankel or a d-Hankel matrix with entries in any ring or field as integers, real numbers or complex numbers.

Let K be a field and $K[X]$ the polynomial ring $K[x_1, \ldots, x_n]$. By a generic Hankel matrix we mean a matrix $X = (y_{ab})$ with $y_{ab} = x_{a+b-1}$. For $j = 1, \ldots, n$ we denote by $X(j, 1, n)$ the $j \times (n+1-j)$ Hankel matrix with entries $x_1, \ldots, x_n$, that is

$$X(j,1,n) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-j+1} \\ x_2 & x_3 & x_4 & \cdots & x_{n-j+2} \\ x_3 & x_4 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_j & x_{j+1} & \cdots & \cdots & x_n \end{bmatrix}.$$

In this matrix, for any $i = 2, \ldots, j$, the ith row is obtained by shifting the $(i-1)$th row one place to the left and inserting a new indeterminate at the end. Now, let $X(j, d, n)$ be the matrix (y_{ab}) with $y_{ab} = x_{d(a-1)+b}$, that is, for $i = 2, \ldots, j$, the ith row is obtained by shifting the $(i-1)$th row d places to the left and inserting d new indeterminates at the end in right. This matrix is called a generalized Hankel matrix or a d-Hankel matrix (see [4]).

The following result is well known. For example, see J. Watanabe [6] or A. Conca [2].

Lemma 1.1 *For $t < j$, every t-minor of $X(j, 1, n)$ is a linear combination of t-minors of $X(j-1, 1, n)$, and every t-minor of $X(j-1, 1, n)$ is a linear combination of t-minors of $X(j, 1, n)$.*

Our aim is to generalize the above result for any d-Hankel matrix.

2 Main Results

Let $X(j, d, n)$ be a d-Hankel matrix with j rows and $n-(j-1)d$ columns, that is:

$$X(j,d,n) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-(j-1)d} \\ x_{d+1} & x_{d+2} & x_{d+3} & \cdots & x_{d+n-(j-1)d} \\ x_{2d+1} & x_{2d+2} & x_{2d+3} & \cdots & x_{2d+n-(j-1)d} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{(j-1)d+1} & x_{(j-1)d+2} & x_{(j-1)d+3} & \cdots & x_n \end{bmatrix}.$$

First, we rearrange the columns:

$$\begin{bmatrix} x_1 & x_{d+1} & x_{2d+1} & \cdots & x_{r_1d+1} & x_2 & x_{d+2} & \cdots & x_{r_dd} \\ x_{d+1} & x_{2d+1} & x_{3d+1} & \cdots & x_{(r_1+1)d+1} & x_{d+2} & x_{2d+2} & \cdots & x_{(r_d+1)d} \\ x_{2d+1} & x_{3d+1} & x_{4d+1} & \cdots & x_{(r_1+2)d+1} & x_{2d+2} & x_{3d+2} & \cdots & x_{(r_d+2)d} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{(j-1)d+1} & x_{jd+1} & x_{(j+1)d+1} & \cdots & x_{(r_1+j-1)d+1} & x_{(j-1)d+2} & x_{jd+2} & \cdots & x_{(r_d+j-1)d} \end{bmatrix}.$$

Where, for all $i, 1 \leq i \leq d$, r_i is the largest integer such that $r_id + i \leq n-(j-1)d$. With this rearrangement, we obtain a matrix with d blocks, each of them an ordinary Hankel matrix. It is clear that the difference between two different m_is is at most 1. Let $N = \max\{r_i : i = 1, \ldots, d\}$. Renaming entries of the above matrix, we have the matrix $H(j, d, N)$:

$$\left[\begin{array}{cccc|c|cccc} x_{11} & x_{12} & \cdots & x_{1(r_1+1)} & & x_{d1} & x_{d2} & \cdots & x_{d(r_d+1)} \\ x_{12} & x_{13} & \cdots & x_{1(r_1+2)} & & x_{d2} & x_{d3} & \cdots & x_{d(r_d+2)} \\ x_{13} & x_{14} & \cdots & x_{1(r_1+3)} & \cdots\cdots & x_{d3} & x_{d4} & \cdots & x_{d(r_d+3)} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ x_{1j} & x_{1(j+1)} & \cdots & x_{1(r_1+j)} & & x_{dj} & x_{d(j+1)} & \cdots & x_{d(r_d+j)} \end{array}\right].$$

A typical t-minor $[a|b]$ of $H(j, d, N)$, for a and b, some integer sequences with t entries such that $1 \leq a \leq j$ and $1 \leq b \leq n-(j-1)d$, has the following form

$$[a_1, a_2, \ldots, a_t|b_{11}, \ldots, b_{1t_1}, \ldots, b_{d1}, \ldots, b_{dt_d}] = \det[x_{a_ib_{kl}}],$$

where, $t = \sum_{i=1}^{d} t_i$, and b_{ij} is a column in the ith block, that is, $1 \leq b_{ij} \leq r_i + 1$. If $b_{it_i} + 1$ be a column in the ith block, for all i, and $a_t + 1 \leq j$, clearly, we have:

$$\begin{aligned} &[a_1+1, a_2+1, \ldots, a_t+1|b_{11}, \ldots, b_{1t_1}, \ldots, b_{d1}, \ldots, b_{dt_d}] = \\ &[a_1, a_2, \ldots, a_t|b_{11}+1, \ldots, b_{1t_1}+1, \ldots, b_{d1}+1, \ldots, b_{dt_d}+1]. \end{aligned} \tag{1}$$

Let $\alpha = \alpha_1, \ldots, \alpha_t$ be a sequence of positive integers, and $I \subset \{1, \cdots, t\}$. We mean by $\alpha + e(I)$ the sequence $\alpha_1 + \delta_1, \cdots, \alpha_t + \delta_t$, where $\delta_i = 1$ if $i \in I$ and $\delta_i = 0$ otherwise. Define α_I the subsequence of α consisting of those with indices in I and $\alpha^{\hat{I}}$ its complement.

Lemma 2.1 *Let $\alpha = \alpha_1, \cdots, \alpha_t$ and $\beta = \beta_1, \cdots, \beta_t$ be strictly increasing sequences of positive integers, such that $1 \leq \alpha \leq j$ and $1 \leq \beta \leq n-(j-1)d$. If $t \leq j$, then for any k, $1 \leq k \leq t$, one has*

$$\sum_{\substack{I \subset \{1, \cdots, t\} \\ |I| = k}} [\alpha + e(I)|\beta] = \sum_{\substack{J \subset \{1, \cdots, t\} \\ |J| = k}} [\alpha|\beta + e(I)] \tag{2}$$

with the conditions just before the equation (1).

Proof. Expanding the minor $[\alpha + e(I)|\beta]$ with respect to the rows with indices in I and expanding the minor $[\alpha|\beta + e(J)]$ with respect to the columns with indices in J, one has:

$$\begin{aligned}\sum_I [\alpha + e(I)|\beta] &= \sum_I \sum_J (-1)^I (-1)^J [\alpha_I + 1|\beta_J][\hat{\alpha^I}|\hat{\beta^J}] \\ &= \sum_J \sum_I (-1)^J (-1)^I [\alpha_I|\beta_J + 1][\hat{\alpha^I}|\hat{\beta^J}] \\ &= \sum_J [\alpha|\beta + e(J)].\end{aligned}$$

Theorem 2.2 *Let $t < j$ and $t \le r_i + 1$, for all $i = 1, \ldots, d$. Then the set of linear combinations of all t-minors of $H(j, d, N)$ is equal to the set of linear combinations of all t-minors of $H(j-1, d, N)$.*

Proof. Let $[a|b] = [a_1, \ldots, a_t|b]$ be a t-minor of $H(j, d, N)$. If $a_t < j$, then $[a|b]$ is already a t-minor of $H(j-1, d, N)$. If $a_t = j$, then let h be the smallest integer such that $a_h = j + h - t$. Applying Lemma 2.1 for the sequence $\alpha = a_1, \ldots, a_{h-1}, a_h - 1, \ldots, a_t - 1$, $\beta = b$ and $k = t - h + 1$, the minor $[a|b]$ can be written as a linear combination of t-minors which are either in $H(j-1, d, N)$ or they are in $H(j, d, N)$ but with a bigger h. By induction on $t - h$ we obtain the assertion in one side.

Now, let $[a|b] = [a|b_{11}, \ldots, b_{1t_1}, \ldots, b_{d1}, \ldots, b_{dt_d}]$ be a t-minor of $H(j-1, d, N)$. If none of b_{it_i} are the last column of their blocks, that is, for every $1 \le i \le d$, $b_{it_i} < r_i + 2$, then $[a|b]$ can be regarded as a t-minor of $H(j, d, N)$ with the same value.

Suppose that for some i, $b_{it_i} = r_i + 2$. Let $\mathsf{M} = \{i|b_{it_i} = r_i + 2, 1 \le i \le d\}$, and $\mathsf{N} = \{i|b_{it_i} = r_i + 1, 1 \le i \le d\}$. For any $i \in \mathsf{M}$, let m_i be the smallest integer such that $b_{i(m_i)} = r_i + 2 + m_i - t_i$. Also for any $i \in \mathsf{N}$ define n_i to be the smallest integer such that $b_{i(n_i)} = r_i + 1 + n_i - t_i$. Let $m = \sum_{i \in \mathsf{M}} m_i$, $n = \sum_{i \in \mathsf{N}} n_i$, $\mathsf{m} = |\mathsf{M}|$, and $\mathsf{n} = |\mathsf{N}|$.

We prove this part of the assertion by induction on the number $m+n$. Without loss of generality, we assume that $\mathsf{M}=\{1, \ldots, \mathsf{m}\}$ and $\mathsf{N}=\{\mathsf{m}+1, \ldots, \mathsf{m}+\mathsf{n}\}$. Let β be the following sequence of integers:

$$\begin{aligned}&b_{11}, \ldots, b_{1(m_1-1)}, b_{1m_1} - 1, \ldots, b_{1t_1} - 1, \ldots, \\ &b_{\mathsf{m}1}, \ldots, b_{\mathsf{m}(m_\mathsf{m}-1)}, b_{\mathsf{m}m_\mathsf{m}} - 1, \ldots, b_{\mathsf{m}t_\mathsf{m}} - 1, \ldots, \\ &b_{(\mathsf{m}+1)1}, \ldots, b_{(\mathsf{m}+1)(n_1-1)}, b_{(\mathsf{m}+1)n_1} - 1, \ldots, b_{(\mathsf{m}+1)t_{\mathsf{m}+1}} - 1, \ldots, \\ &b_{(\mathsf{m}+\mathsf{n})1}, \ldots, b_{(\mathsf{m}+\mathsf{n})(n_\mathsf{n}-1)}, b_{(\mathsf{m}+\mathsf{n})n_\mathsf{n}} - 1, \ldots, b_{(\mathsf{m}+\mathsf{n})t_{\mathsf{m}+\mathsf{n}}} - 1, \ldots, \\ &b_{(\mathsf{m}+\mathsf{n}+1)1}, \ldots, b_{(\mathsf{m}+\mathsf{n}+1)t_{\mathsf{m}+\mathsf{n}+1}}, \ldots, b_{d1}, \ldots, b_{dt_d}.\end{aligned}$$

Now apply Lemma 2.1 for $\alpha = a$, β, and $k = \mathsf{m}+\mathsf{n}$. In the equation (2), all minors in right hand side are in $H(j, d, N)$, and for $I = \{1, \ldots, \mathsf{m}+\mathsf{n}\}$ we have

$[\alpha|\beta + e(I)] = [\alpha|\beta]$. All others in left are minors where the related number $m+n$ is strictly less than the number for $[a|b]$ and using induction we have done.

Applying the above theorem $j - t$ times, we finally obtain a matrix with t rows and then any t-minor is a maximal minor of the matrix.

Remark i) The condition $t \leq r_i + 1$ in Theorem 2.2, is necessary: one can take a $(r_i + 2)$-minor in $H(j-1, d, N)$ consisting of all columns of the ith block, which is not a linear combination of $(r_i + 2)$-minors of $H(j, d, N)$.
ii) Note that the Theorem is valid for concatenation of Hankel matrices of size $j \times r_i + 1$ for arbitrary r_i's, provided $t \leq r_i + 1$ for all $i = 1, \ldots, d$.

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