UNIVERSAL ENVELOPING ALGEBRAS OF THE FOUR-DIMENSIONAL MALCEV ALGEBRA

M. R. BREMNER, I. R. HENTZEL, L. A. PERESI, AND H. USEFI

Dedicated to Professor Ivan P. Shestakov, and presented at the conference on "Algebras, Representations and Applications", in honor of his sixtieth birthday, held at Maresias, São Paulo, Brazil from August 26 to September 1, 2007.

Abstract. We determine structure constants for the universal nonassociative enveloping algebra $U(\mathbb{M})$ of the four-dimensional non-Lie Malcev algebra $\mathbb{M}$ by constructing a representation of $U(\mathbb{M})$ by differential operators on the polynomial algebra $P(\mathbb{M})$. The structure constants for $U(\mathbb{M})$ involve the Stirling numbers of the second kind. This work is based on the recent theorem of Pérez-Izquierdo and Shestakov which generalizes the Poincaré-Birkhoff-Witt theorem from Lie algebras to Malcev algebras. We use our results for $U(\mathbb{M})$ to determine structure constants for the universal alternative enveloping algebra $A(\mathbb{M}) = U(\mathbb{M})/I(\mathbb{M})$ where $I(\mathbb{M})$ is the alternator ideal of $U(\mathbb{M})$. The structure constants for $A(\mathbb{M})$ were obtained earlier by Shestakov using different methods.

1. Introduction

A Malcev algebra M over a field $\mathbb{F}$ is a vector space with a bilinear product $M \times M \to M$ denoted $(x, y) \mapsto [x, y]$, satisfying the anticommutative identity $[x, x] = 0$ and the Malcev identity $[J(x, y, z), x] = J(x, y, [x, z])$, where $J(x, y, z) = [[x, y], z] + [[y, z], x] + [[z, x], y]$. These two identities hold for the commutator $[x, y] = xy - yx$ in any alternative algebra. Basic references on Malcev algebras are [1, 2, 3, 4, 6].

The Poincaré-Birkhoff-Witt (PBW) theorem constructs, for any Lie algebra L, a universal associative enveloping algebra $U(L)$ together with an injective Lie algebra morphism $\iota \colon L \to U(L)^-$; thus L is isomorphic to a subalgebra of the commutator algebra of an associative algebra. It is an open problem whether every Malcev algebra is special (isomorphic to a subalgebra of the commutator algebra of an alternative algebra); see Shestakov [7, 8, 9], Shestakov and Zhukavets [11, 12, 13, 14]. A solution to a closely related problem was given a

2000 *Mathematics Subject Classification.* Primary 17D10; Secondary 17D05.

few years ago by Pérez-Izquierdo and Shestakov [5]: they constructed universal nonassociative enveloping algebras for Malcev algebras.

In dimension 4, there is (up to isomorphism) a unique non-Lie Malcev algebra over any field of characteristic $\neq 2, 3$. This algebra is solvable; its structure constants appear in Table 1. We write $\mathbb{M}$ for this algebra, and M for an arbitrary Malcev algebra. In this paper we determine: (1) explicit structure constants for the universal nonassociative enveloping algebra $U(\mathbb{M})$; (2) a finite set of generators for the alternator ideal $I(\mathbb{M}) \subset U(\mathbb{M})$; (3) explicit structure constants for the universal alternative enveloping algebra $A(\mathbb{M}) = U(\mathbb{M})/I(\mathbb{M})$. Shestakov [8, Example 1] found the structure constants for $A(\mathbb{M})$ as an application of Malcev Poisson algebras. Shestakov and Zhelyabin [10] proved that if M is finite dimensional and semisimple then $U(M)$ is a free module over its center and that the center is isomorphic to a polynomial algebra on n variables where n is the dimension of the Cartan subalgebra; they also calculate the center of $U(M)$ for several Malcev algebras of small dimension. In the case $M = \mathbb{M}$, the center can be obtained as a corollary to our structure constants for $U(\mathbb{M})$.

Table 1. The four-dimensional Malcev algebra $\mathbb{M}$

$[\,,]$	a	b	c	d
a	0	$-b$	$-c$	d
b	b	0	$2d$	0
c	c	$-2d$	0	0
d	$-d$	0	0	0

2. Preliminary results

All multilinear structures are over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2, 3$.

Definition 2.1. The **generalized alternative nucleus** of a nonassociative algebra A is

$$N_{\text{alt}}(A) = \big\{\, a \in A \,|\, (a, x, y) = -(x, a, y) = (x, y, a), \,\forall\, x, y \in A \,\big\},$$

where the **associator** is $(x, y, z) = (xy)z - x(yz)$.

Lemma 2.2. *In general $N_{\text{alt}}(A)$ is not a subalgebra of A, but it is a subalgebra of A^- and is a Malcev algebra.*

Theorem 2.3 (Pérez-Izquierdo and Shestakov)**.** *For every Malcev algebra M there is a universal nonassociative enveloping algebra $U(M)$ and an injective morphism $\iota\colon M \to U(M)^-$ with $\iota(M) \subseteq N_{alt}(U(M))$.*

Let $F(M)$ be the unital free nonassociative algebra on a basis of M. Let $R(M)$ be the ideal generated by the elements $ab - ba - [a, b]$, $(a, x, y) + (x, a, y)$, $(x, a, y) + (x, y, a)$ for all $a, b \in M$, $x, y \in F(M)$. Define $U(M) = F(M)/R(M)$, and the mapping $\iota\colon M \to U(M)$ by $a \mapsto \iota(a) = \overline{a} = a + R(M)$. Since ι is injective, we identify M with $\iota(M) \subseteq N_{\mathrm{alt}}(U(M))$. Let $B = \{a_i \,|\, i \in \mathcal{I}\}$ be a basis of M with $<$ a total order on the index set $\mathcal{I}$. Define $\Omega = \{\,(i_1, \dots, i_n) \,|\, i_1 \le \cdots \le i_n\,\}$. The empty tuple $\emptyset$ ($n = 0$) gives $\overline{a}_\emptyset = 1 \in U(M)$. The n-tuple $I = (i_1, \dots, i_n) \in \Omega$ ($n \ge 1$) defines a left-tapped monomial $\overline{a}_I = \overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots))$ of degree $|\overline{a}_I| = n$. The set $\{\,\overline{a}_I \,|\, I \in \Omega\,\}$ is a basis of $U(M)$. For details, see Pérez-Izquierdo and Shestakov [5].

For any $f, g \in M$ and $y \in U(M)$, since $f, g \in N_{\mathrm{alt}}(U(M))$ we obtain

$$(f, g, y) = \tfrac{1}{6}[[y, f], g] - \tfrac{1}{6}[[y, g], f] - \tfrac{1}{6}[[y, [f, g]].$$

This equation implies the next three lemmas, which are implicit in [5]. We first compute $[x, f]$ in $U(M)$; for $|x| = 1$ we use the bracket in M.

Lemma 2.4. *Let x be a basis monomial of $U(M)$ with $|x| \ge 2$, and let f be an element of M. Write $x = gy$ with $g \in M$. Then*

$$[x, f] = [gy, f] = [g, f]y + g[y, f] + \tfrac{1}{2}[[y, f], g] - \tfrac{1}{2}[[y, g], f] - \tfrac{1}{2}[y, [f, g]].$$

We next compute fx in $U(M)$; for $|x| = 1$ we have two cases: if $f \le x$ in the ordered basis, then fx is a basis monomial; otherwise, $fx = xf + [f, x]$ where $[f, x] \in M$.

Lemma 2.5. *Let x be a basis monomial of $U(M)$ with $|x| \ge 2$, and let f be an element of M. Write $x = gy$ with $g \in M$. Then*

$$fx = f(gy) = g(fy) + [f, g]y - \tfrac{1}{3}[[y, f], g] + \tfrac{1}{3}[[y, g], f] + \tfrac{1}{3}[y, [f, g]].$$

We finally compute yz in $U(M)$; for $|y| = 1$ we use Lemma 2.5.

Lemma 2.6. *Let y and z be basis monomials of $U(M)$ with $|y| \ge 2$. Write $y = fx$ with $f \in M$. Then*

$$yz = (fx)z = 2f(xz) - x(fz) - x[z, f] + [xz, f].$$

Expansion in the free nonassociative algebra establishes the identity

$$(pq, r, s) - (p, qr, s) + (p, q, rs) = p(q, r, s) + (p, q, r)s.$$

From this equation the next lemma easily follows.

Lemma 2.7. *For all $g \in M$ and $x \in U(M)$ we have*

$$(g^i, g, x) = (g^i, x, g) = (g, g^i, x) = (g, x, g^i) = (x, g^i, g) = (x, g, g^i) = 0.$$

From this, induction gives $(g^j, g^i, x) = 0$ and hence $[g^k x, g] = g^k[x, g]$.

The algebra $\mathbb{M}$ has solvable Lie subalgebras with bases $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, and a nilpotent Lie subalgebra with basis $\{b,c,d\}$. The next two lemmas are standard computations in enveloping algebras.

Lemma 2.8. *For $e \in \{b,c\}$ these equations hold in $U(\mathbb{M})$:*

$$(a^i e^j)(a^k e^\ell) = a^i (a{+}j)^k e^{j+\ell}, \quad (a^i d^j)(a^k d^\ell) = a^i (a{-}j)^k d^{j+\ell}.$$

Lemma 2.9. *These equations hold in $U(\mathbb{M})$:*

$$(b^i c^j d^k)(b^\ell c^m d^n) = \sum_{h=0}^{\ell} (-1)^h 2^h \binom{\ell}{h} \frac{j!}{(j{-}h)!} b^{i+\ell-h} c^{j+m-h} d^{k+n+h},$$

$$[b^i c^j d^k, b] = -2j b^i c^{j-1} d^{k+1}, \;\; [b^i c^j d^k, c] = 2i b^{i-1} c^j d^{k+1}, \;\; [b^i c^j d^k, d] = 0.$$

The following representation will play an important role in our computation of the structure constants for $U(\mathbb{M})$.

Definition 2.10. Let M be a Malcev algebra, and let $P(M)$ be the polynomial algebra on a basis of M. By Theorem 2.3 we have a linear isomorphism $\phi\colon U(M) \to P(M)$ defined by

$$\overline{a}_{i_1}(\cdots(\overline{a}_{i_{n-2}}(\overline{a}_{i_{n-1}}\overline{a}_{i_n}))\cdots) \longmapsto a_{i_1} \cdots a_{i_{n-2}} a_{i_{n-1}} a_{i_n}.$$

For $x \in U(M)$, $f \in P(M)$ we define the **right bracket operator** ρ and the **left multiplication operator** L as follows:

$$\rho(x)(f) = \phi\big([\phi^{-1}(f), x]\big), \qquad L(x)(f) = \phi\big(x\phi^{-1}(f)\big).$$

Thus $\rho(x)$ (respectively $L(x)$) is the operator on $P(M)$ induced by the mapping $y \mapsto [y,x]$ (respectively $y \mapsto xy$) in $U(M)$. We also have the **right multiplication operator** $R(x) = \rho(x) + L(x)$.

3. Representation of $\mathbb{M}$ by differential operators

Definition 3.1. We have these operators on $P(\mathbb{M})$: I is the **identity**; M_x is **multiplication** by $x \in \{a,b,c,d\}$; D_x is **differentiation** with respect to $x \in \{a,b,c,d\}$; S is the **shift** $a \mapsto a{+}1$: $S(a^i b^j c^k d^\ell) = (a{+}1)^i b^j c^k d^\ell$. Since S is invertible, S^t is defined for all $t \in \mathbb{Z}$.

In this Section we determine $\rho(x)$ and $L(x)$ for $x \in \{a,b,c,d\}$ as differential operators on $P(\mathbb{M})$. We summarize our results in Table 2.

Lemma 3.2. *For $x, y \in \{a,b,c,d\}$ we have*

$$[D_x, M_x] = I, \qquad [D_x, M_y] = 0 \ (x \ne y), \;\; [D_x, D_y] = 0, \;\; [M_x, M_y] = 0,$$

$$[M_a, S] = -S, \qquad [M_x, S] = 0 \ (x \ne a), \quad [D_x, S] = 0, \;\; [D_x, S^{-1}] = 0,$$

$$[M_a, S^{-1}] = S^{-1}, \;\; [M_x, S^{-1}] = 0 \ (x \ne a).$$

Proof. These follow easily from Definition 3.1. □

TABLE 2. Differential operators $\rho(x)$ and $L(x)$ on $P(\mathbb{M})$

x	$\rho(x)$	$L(x)$
a	$M_bD_b + M_cD_c - M_dD_d - 3M_dD_bD_c$	M_a
b	$(I-S)M_b + (S-I-2S^{-1})\,M_dD_c$	$SM_b + (S^{-1}-S)\,M_dD_c$
c	$(I-S)M_c + (S-I+2S^{-1})\,M_dD_b$	$SM_c - (S^{-1}+S)\,M_dD_b$
d	$(I-S^{-1})\,M_d$	$S^{-1}M_d$

Lemma 3.3. *We have* $[b^nc^pd^q, a] = (n+p-q)b^nc^pd^q - 3npb^{n-1}c^{p-1}d^{q+1}$.

Proof. Induction on n; the basis $n = 0$ is $[c^pd^q, a] = (p-q)c^pd^q$, which follows since a, c, d span a Lie subalgebra of $\mathbb{M}$. We now let $n \geq 0$ and use Lemma 2.4 with $f = a$, $g = b$; we see that $[b^{n+1}c^pd^q, a]$ equals

$$[ba]b^nc^pd^q + b[b^nc^pd^q, a] + \tfrac{1}{2}\big([[b^nc^pd^q, a], b] - [[b^nc^pd^q, b], a] - [b^nc^pd^q, [ab]]\big).$$

We apply Lemma 2.9 to the right side:

$$b^{n+1}c^pd^q + b[b^nc^pd^q, a] + \tfrac{1}{2}[[b^nc^pd^q, a], b] + p[b^nc^{p-1}d^{q+1}, a] - pb^nc^{p-1}d^{q+1}.$$

The inductive hypothesis gives

$$\begin{aligned} &b^{n+1}c^pd^q + (n+p-q)b^{n+1}c^pd^q - 3npb^nc^{p-1}d^{q+1} + \tfrac{1}{2}(n+p-q)[b^nc^pd^q, b] \\ &- \tfrac{3}{2}np[b^{n-1}c^{p-1}d^{q+1}, b] + (n+p-q-2)pb^nc^{p-1}d^{q+1} \\ &- 3np(p-1)b^{n-1}c^{p-2}d^{q+2} - pb^nc^{p-1}d^{q+1}. \end{aligned}$$

We use Lemma 2.9 again to get

$$\begin{aligned} &b^{n+1}c^pd^q + (n+p-q)b^{n+1}c^pd^q - 3np\,b^nc^{p-1}d^{q+1} - (n+p-q)pb^nc^{p-1}d^{q+1} \\ &+ 3np(p-1)b^{n-1}c^{p-2}d^{q+2} + (n+p-q-2)pb^nc^{p-1}d^{q+1} \\ &- 3np(p-1)b^{n-1}c^{p-2}d^{q+2} - pb^nc^{p-1}d^{q+1}. \end{aligned}$$

Combining terms gives $(n+1+p-q)b^{n+1}c^pd^q - 3(n+1)pb^nc^{p-1}d^{q+1}$. □

Lemma 3.4. *We have*

$$\rho(a) = M_bD_b + M_cD_c - M_dD_d - 3M_dD_bD_c, \qquad L(a) = M_a.$$

Proof. Lemma 2.7 gives $[a^mb^nc^pd^q, a] = a^m[b^nc^pd^q, a]$, and now Lemma 3.3 gives the formula for $\rho(a)$. The formula for $L(a)$ is clear. □

Lemma 3.5. *We have*

$$\rho(b) = \big(I-S\big)M_b + \big(S-I-2S^{-1}\big)M_dD_c, \;\; L(b) = SM_b + \big(S^{-1}-S\big)M_dD_c.$$

Proof. Induction on m where $y = a^mb^nc^pd^q$. We prove the formulas together, since each requires the inductive hypothesis of the other. The basis $m = 0$ for $\rho(b)$ is Lemma 2.9, and for $L(b)$ it is clear. We assume both formulas for $m \geq 0$. Lemma 2.4 ($f = b$, $g = a$) gives

$$\begin{aligned}
\rho(b)(ay) &= -by + a[y,b] + \tfrac{1}{2}\Big([[y,b],a] - [[y,a],b] - [y,b]\Big) \\
&= \Big(- L(b) + M_a\rho(b) + \tfrac{1}{2}[\rho(a),\rho(b)] - \tfrac{1}{2}\rho(b)\Big)(y).
\end{aligned}$$

The inductive hypothesis for $\rho(b)$, Lemma 3.4 and Lemma 3.2 give

$$[\rho(a),\rho(b)](y) = \Big((I - S)M_b + (S - I + 4S^{-1})M_dD_c\Big)(y).$$

Combining this with both inductive hypotheses we get

$$\begin{aligned}
\rho(b)(ay) &= -\Big(SM_b + (S^{-1}{-}S)M_dD_c\Big)(y) \\
&\quad + M_a\Big((I{-}S)M_b + (S{-}I{-}2S^{-1})M_dD_c\Big)(y) \\
&\quad + \tfrac{1}{2}\Big((I{-}S)M_b + (S{-}I{+}4S^{-1})M_dD_c\Big)(y) \\
&\quad - \tfrac{1}{2}\Big((I{-}S)M_b + (S{-}I{-}2S^{-1})M_dD_c\Big)(y) \\
&= \Big(M_a - (M_a{+}I)S\Big)M_b(y) \\
&\quad + \Big((M_a{+}I)S - M_a - 2(M_a{-}I)S^{-1}\Big)M_dD_c(y) \\
&= (I{-}S)M_b(ay) + (S{-}I{-}2S^{-1})M_dD_c(ay),
\end{aligned}$$

which completes the proof for $\rho(b)$. Lemma 2.5 with $f = b$, $g = a$ gives

$$\begin{aligned}
L(b)(ay) &= (a{+}1)(by) + \tfrac{1}{3}\Big([[y,a],b] - [[y,b],a] + [y,b]\Big) \\
&= \big((a{+}1)L(b) - \tfrac{1}{3}[\rho(a),\rho(b)] + \tfrac{1}{3}\rho(b)\big)\,(y).
\end{aligned}$$

Using the inductive hypotheses for $L(b)$ and $\rho(b)$ we get

$$\begin{aligned}
L(b)(ay) &= (a{+}1)SM_b(y) + (a{+}1)(S^{-1}{-}S)M_dD_c(y) \\
&\quad - \tfrac{1}{3}\Big((I{-}S)M_b + (S{-}I{+}4S^{-1})M_dD_c\Big)(y) \\
&\quad + \tfrac{1}{3}\Big((I{-}S)M_b + (S{-}I{-}2S^{-1})M_dD_c\Big)(y) \\
&= (a{+}1)SM_b(y) + \Big((a{-}1)S^{-1} - (a{+}1)S\Big)M_dD_c(y) \\
&= SM_b(ay) + (S^{-1}{-}S)M_dD_c(ay),
\end{aligned}$$

which completes the proof for $L(b)$. □

Lemma 3.6. *We have*

$\rho(c) = \big(I{-}S\big)M_c{+}\big(S{-}I{+}2S^{-1}\big)M_dD_b, \;\; L(c) = SM_c{-}\big(S{+}S^{-1}\big)M_dD_b.$

Proof. Similar to the proof of Lemma 3.5. □

Lemma 3.7. *We have* $\rho(d) = \big(I - S^{-1}\big)M_d$ *and* $L(d) = S^{-1}M_d$.

Proof. Induction on m where $y = a^m b^n c^p d^q$. We prove both formulas together. The basis $m = 0$ is Lemma 2.9. We assume both formulas for $m \geq 0$. Lemma 2.4 with $f = d$, $g = a$ gives

$$\begin{aligned}\rho(d)(ay) &= dy + a[y,d] + \tfrac{1}{2}\Big([[y,d],a] - [[y,a],d] + [y,d]\Big)\\ &= \big(L(d) + M_a\rho(d) + \tfrac{1}{2}[\rho(a),\rho(d)] + \tfrac{1}{2}\rho(d)\big)(y).\end{aligned}$$

The inductive hypothesis gives $[\rho(a),\rho(d)](y) = -\rho(d)(y)$ and so

$$\rho(d)(ay) = \Big(L(d) + M_a\rho(d)\Big)(y) = (I - S^{-1})M_d(ay),$$

which completes the proof for $\rho(d)$. Lemma 2.5 with $f = d$, $g = a$ gives

$$\begin{aligned}L(d)(ay) &= a(dy) - dy - \tfrac{1}{3}[[y,d],a] + \tfrac{1}{3}[[y,a],d] - \tfrac{1}{3}[y,d]\\ &= \big(M_a L(d) - L(d) + \tfrac{1}{3}[\rho(d),\rho(a)] - \tfrac{1}{3}\rho(d)\big)(y)\\ &= \Big(M_a L(d) - L(d)\Big)(y) = (M_a - I)S^{-1}M_d(y) = S^{-1}M_d(ay),\end{aligned}$$

which completes the proof for $L(d)$. □

4. Representation of $U(\mathbb{M})$ by differential operators

In this Section we determine $L(x)$ for $x = a^i b^j c^k d^\ell$ as a differential operator on $P(\mathbb{M})$. We often use the facts that linear operators E, F, G satisfy $[E, FG] = [E,F]G + F[E,G]$, and that if $[[E,F],F] = 0$ then $[E, F^k] = k[E,F]F^{k-1}$ for every $k \geq 1$.

Since c, d span an Abelian Lie subalgebra $\mathbb{A} \subset \mathbb{M}$, associativity gives $L(c^k d^\ell) = L(c)^k L(d)^\ell$ on $U(\mathbb{A})$; this is also true on $U(\mathbb{M})$.

Lemma 4.1. *In* $U(\mathbb{M})$ *we have* $L(c^k d^\ell) = L(c)^k L(d)^\ell$.

Proof. We first prove $L(d^\ell) = L(d)^\ell$ by induction. For $\ell \geq 1$ we get

$$(dd^\ell)(a^m b^n c^p d^q) = (d, d^\ell, a^m b^n c^p d^q) + d\Big((d^\ell)(a^m b^n c^p d^q)\Big).$$

The associator is zero by Lemma 2.7. We now use induction on k. Lemma 2.6 with $f = c$, $x = c^k d^\ell$ gives

$$(c^{k+1}d^\ell)z = 2c((c^k d^\ell)z) - (c^k d^\ell)(cz) - (c^k d^\ell)[z,c] + [(c^k d^\ell)z, c],$$

which can be written as

$$L(c^{k+1}d^\ell) = L(c)L(c^k d^\ell) + [L(c), L(c^k d^\ell)] + [\rho(c), L(c^k d^\ell)].$$

The inductive hypothesis gives

$$[\rho(c), L(c^k d^\ell)] = L(c)^k[\rho(c), L(d)^\ell] + [\rho(c), L(c)^k]L(d)^\ell = 0,$$

and similarly $[L(c), L(c^k d^\ell)] = 0$. □

Since b, c, d span a nilpotent Lie subalgebra $\mathbb{N} \subset \mathbb{M}$, associativity gives $L(b^j c^k d^\ell) = L(b)^j L(c)^k L(d)^\ell$ on $U(\mathbb{N})$; this is not true on $U(\mathbb{M})$.

Lemma 4.2. *In $U(\mathbb{M})$ the operator $L(b^j c^k d^\ell)$ equals*

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} (-1)^{\alpha-\beta} \alpha! \binom{\alpha}{\beta} \binom{j}{\alpha} \binom{k}{\alpha} S^{-\beta} L(b)^{j-\alpha} L(c)^{k-\alpha} M_d^\alpha L(d)^\ell.$$

Proof. Induction on j; the basis is Lemma 4.1. Lemma 2.6 with $f = b$, $x = b^j c^k d^\ell$, $z = a^m b^n c^p d^q$ gives

$$\begin{aligned}(b^{j+1} c^k d^\ell)(a^m b^n c^p d^q) &= 2b(xz) - x(bz) - x[z, b] + [xz, b] \\ &= 2L(b)L(x)z - L(x)L(b)z - L(x)\rho(b)z + \rho(b)L(x)z \\ &= L(b)L(x)z + [L(b), L(x)]z + [\rho(b), L(x)]z \\ &= L(b)L(x)z + [R(b), L(x)]z.\end{aligned}$$

Induction and $[R(b), L(b)] = [R(b), M_d] = [R(b), L(d)] = 0$ show that $[R(b), L(x)]$ equals

$$\sum_{\alpha=0}^{\min(j,k)} \alpha! \binom{j}{\alpha} \binom{k}{\alpha} \left(S^{-1} - I\right)^\alpha L(b)^{j-\alpha} \left[R(b), L(c)^{k-\alpha}\right] M_d^\alpha L(d)^\ell =$$

$$\sum_{\alpha=0}^{\min(j,k)} \alpha! \binom{j}{\alpha} (k-\alpha) \binom{k}{\alpha} \left(S^{-1} - I\right)^{\alpha+1} L(b)^{j-\alpha} L(c)^{k-\alpha-1} M_d^{\alpha+1} L(d)^\ell.$$

Replacing α by $\alpha-1$ gives

$$\sum_{\alpha=1}^{\min(j+1,k)} \alpha! \binom{j}{\alpha-1} \binom{k}{\alpha} \left(S^{-1} - I\right)^\alpha L(b)^{j+1-\alpha} L(c)^{k-\alpha} M_d^\alpha L(d)^\ell.$$

We use Pascal's identity $\binom{j}{\alpha} + \binom{j}{\alpha-1} = \binom{j+1}{\alpha}$ to combine $L(b)L(x)$ and $[R(b), L(x)]$, and obtain this formula for $L(b^{j+1} c^k d^\ell)$:

$$\sum_{\alpha=0}^{\min(j+1,k)} \alpha! \binom{j+1}{\alpha} \binom{k}{\alpha} \left(S^{-1} - I\right)^\alpha L(b)^{j+1-\alpha} L(c)^{k-\alpha} M_d^\alpha L(d)^\ell.$$

We now expand $(S^{-1} - I)^\alpha$ with the binomial theorem. □

Lemma 4.3. *We have*

$$\begin{aligned}&[R(a), L(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z] = \\ &\quad - (t+v+w+y) L(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z \\ &\qquad - u L(a)^s S^{t-1} L(b)^{u-1} D_b^v D_c^{w+1} L(c)^x M_d^{y+1} L(d)^z\end{aligned}$$

$$+ xL(a)^s S^{t-1} L(b)^u D_b^{v+1} D_c^w L(c)^{x-1} M_d^{y+1} L(d)^z.$$

Proof. Table 2 and Lemma 3.2 give

$$\begin{aligned}
&[R(a), L(a)] = 0, && [R(a), L(b)] = -S^{-1} M_d D_c, \\
&[R(a), L(c)] = S^{-1} M_d D_b, && [R(a), L(d)] = 0, \\
&[R(a), D_b] = -D_b, && [R(a), D_c] = -D_c, \\
&[R(a), M_d] = -M_d, && [R(a), S] = -S.
\end{aligned}$$

From these equations we get

$$\begin{aligned}
&[R(a), L(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z] = \\
&\quad L(a)^s \big[R(a), S^t\big] L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z \\
&\quad + L(a)^s S^t \big[R(a), L(b)^u\big] D_b^v D_c^w L(c)^x M_d^y L(d)^z \\
&\quad + L(a)^s S^t L(b)^u \big[R(a), D_b^v\big] D_c^w L(c)^x M_d^y L(d)^z \\
&\quad + L(a)^s S^t L(b)^u D_b^v \big[R(a), D_c^w\big] L(c)^x M_d^y L(d)^z \\
&\quad + L(a)^s S^t L(b)^u D_b^v D_c^w \big[R(a), L(c)^x\big] M_d^y L(d)^z \\
&\quad + L(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x \big[R(a), M_d^y\big] L(d)^z.
\end{aligned}$$

The right side simplifies to

$$\begin{aligned}
&- tL(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z \\
&\quad - uL(a)^s S^t L(b)^{u-1} S^{-1} D_c M_d D_b^v D_c^w L(c)^x M_d^y L(d)^z \\
&\quad - vL(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z \\
&\quad - wL(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z \\
&\quad + xL(a)^s S^t L(b)^u D_b^v D_c^w L(c)^{x-1} S^{-1} D_b M_d M_d^y L(d)^z \\
&\quad - yL(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z,
\end{aligned}$$

which gives the result. □

Lemma 4.4. *In $U(\mathbb{M})$ the operator $L(a^i b^j c^k d^\ell)$ equals*

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i} \sum_{\delta=0}^{i-\gamma} \sum_{\epsilon=0}^{i-\gamma-\delta} (-1)^{i+\alpha-\beta-\gamma-\delta} \alpha!\delta!\epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} \times$$

$$X_i(\gamma, \delta, \epsilon) L(a)^\gamma S^{-\beta-\delta-\epsilon} L(b)^{j-\alpha-\epsilon} D_b^\delta D_c^\epsilon L(c)^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon} L(d)^\ell,$$

where $X_i(\gamma, \delta, \epsilon)$ is a polynomial in $\alpha-\beta$ satisfying the recurrence

$$X_{i+1}(\gamma, \delta, \epsilon) =$$

$$(\alpha-\beta+\delta+\epsilon) X_i(\gamma, \delta, \epsilon) + X_i(\gamma-1, \delta, \epsilon) + X_i(\gamma, \delta-1, \epsilon) + X_i(\gamma, \delta, \epsilon-1),$$

with $X_0(0,0,0) = 1$ and $X_i(\gamma, \delta, \epsilon) = 0$ unless $0 \le \gamma \le i$, $0 \le \delta \le i-\gamma$, $0 \le \epsilon \le i-\gamma-\delta$.

Proof. Induction on i; the basis $i = 0$ is Lemma 4.2. Lemma 2.6 with $f = a$, $x = a^i b^j c^k d^\ell$, $z = a^m b^n c^p d^q$ gives

$$(a^{i+1} b^j c^k d^l)(a^m b^n c^p d^q) = L(a)L(x)z + [R(a), L(x)]z.$$

Induction and Lemma 4.3 give $[R(a), L(x)] = A + B + C$ where

$$A = -\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i} \sum_{\delta=0}^{i-\gamma} \sum_{\epsilon=0}^{i-\gamma-\delta} (-1)^{i+\alpha-\beta-\gamma-\delta} \times$$
$$(\alpha-\beta+\delta+\epsilon)\alpha!\delta!\epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} X_i(\gamma, \delta, \epsilon) \times$$
$$L(a)^\gamma S^{-\beta-\delta-\epsilon} L(b)^{j-\alpha-\epsilon} D_b^\delta D_c^\epsilon L(c)^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon} L(d)^\ell,$$

$$B = -\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i} \sum_{\delta=0}^{i-\gamma} \sum_{\epsilon=0}^{i-\gamma-\delta} (-1)^{i+\alpha-\beta-\gamma-\delta} \times$$
$$(j-\alpha-\epsilon)\alpha!\delta!\epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} X_i(\gamma, \delta, \epsilon) \times$$
$$L(a)^\gamma S^{-\beta-\delta-\epsilon-1} L(b)^{j-\alpha-\epsilon-1} D_b^\delta D_c^{\epsilon+1} L(c)^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon+1} L(d)^\ell,$$

$$C = \sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i} \sum_{\delta=0}^{i-\gamma} \sum_{\epsilon=0}^{i-\gamma-\delta} (-1)^{i+\alpha-\beta-\gamma-\delta} \times$$
$$(k-\alpha-\delta)\alpha!\delta!\epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} X_i(\gamma, \delta, \epsilon) \times$$
$$L(a)^\gamma S^{-\beta-\delta-\epsilon-1} L(b)^{j-\alpha-\epsilon} D_b^{\delta+1} D_c^\epsilon L(c)^{k-\alpha-\delta-1} M_d^{\alpha+\delta+\epsilon+1} L(d)^\ell.$$

We write $D = L(a)L(x)$ and obtain

$$D = \sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i} \sum_{\delta=0}^{i-\gamma} \sum_{\epsilon=0}^{i-\gamma-\delta} (-1)^{i+\alpha-\beta-\gamma-\delta} \times$$
$$\alpha!\delta!\epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} X_i(\gamma, \delta, \epsilon) \times$$
$$L(a)^{\gamma+1} S^{-\beta-\delta-\epsilon} L(b)^{j-\alpha-\epsilon} D_b^\delta D_c^\epsilon L(c)^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon} L(d)^\ell.$$

In A, we include the term (which is zero) for $\epsilon = i{+}1{-}\gamma{-}\delta$, and absorb the minus sign. In B we replace ϵ by $\epsilon{-}1$, include the term for $\epsilon = 0$, simplify the coefficient using $(j{-}\alpha{-}\epsilon{+}1)(\epsilon{-}1)!\binom{j}{\alpha,\epsilon-1} = \epsilon!\binom{j}{\alpha,\epsilon}$, and absorb the minus sign. In C we replace δ by $\delta{-}1$, include the term for $\delta = 0$, and simplify the coefficient using $(k{-}\alpha{-}\delta{+}1)(\delta{-}1)!\binom{k}{\alpha,\delta-1} = \delta!\binom{k}{\alpha,\delta}$. In D we replace γ by $\gamma{-}1$, and include the term for $\gamma = 0$. We find that $A+B+C+D$ equals

$$\sum_{\alpha=0}^{\min(j,k)} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{i+1} \sum_{\delta=0}^{i+1-\gamma} \sum_{\epsilon=0}^{i+1-\gamma-\delta} (-1)^{i+1+\alpha-\beta-\gamma-\delta} \times$$
$$\alpha!\delta!\epsilon! \binom{\alpha}{\beta} \binom{j}{\alpha, \epsilon} \binom{k}{\alpha, \delta} X_{i+1}(\gamma, \delta, \epsilon) \times$$
$$L(a)^{\gamma} S^{-\beta-\delta-\epsilon} L(b)^{j-\alpha-\epsilon} D_b^{\delta} D_c^{\epsilon} L(c)^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon} L(d)^{\ell},$$

where $X_i(\gamma, \delta, \epsilon)$ satisfies the stated recurrence relation. □

Definition 4.5. The **Stirling numbers of the second kind** are

$$\left\{ {r \atop s} \right\} = \frac{1}{s!} \sum_{t=0}^{s} (-1)^{s-t} \binom{s}{t} t^r.$$

Lemma 4.6. *The unique solution to the recurrence of Lemma 4.4 is*

$$X_i(\gamma, \delta, \epsilon) = \binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i-\gamma-\delta-\epsilon} \binom{i}{\gamma, \zeta} \left\{ {i-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^{\zeta}.$$

Proof. The right side of the recurrence is the sum of these five terms:

$$(\alpha-\beta) X_i(\gamma, \delta, \epsilon) = \binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=1}^{i+1-\gamma-\delta-\epsilon} \binom{i}{\gamma, \zeta-1} \left\{ {i+1-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^{\zeta},$$

$$(\delta+\epsilon) X_i(\gamma, \delta, \epsilon) = \binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i-\gamma-\delta-\epsilon} \binom{i}{\gamma, \zeta} (\delta+\epsilon) \left\{ {i-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^{\zeta},$$

$$X_i(\gamma-1, \delta, \epsilon) = \binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i+1-\gamma-\delta-\epsilon} \binom{i}{\gamma-1, \zeta} \left\{ {i+1-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^{\zeta},$$

$$X_i(\gamma, \delta-1, \epsilon) = \binom{\delta-1+\epsilon}{\epsilon} \sum_{\zeta=0}^{i+1-\gamma-\delta-\epsilon} \binom{i}{\gamma, \zeta} \left\{ {i-\gamma-\zeta \atop \delta-1+\epsilon} \right\} (\alpha-\beta)^{\zeta},$$

$$X_i(\gamma, \delta, \epsilon-1) = \binom{\delta+\epsilon-1}{\epsilon-1} \sum_{\zeta=0}^{i+1-\gamma-\delta-\epsilon} \binom{i}{\gamma, \zeta} \left\{ {i-\gamma-\zeta \atop \delta+\epsilon-1} \right\} (\alpha-\beta)^{\zeta}.$$

Pascal's formula shows that $X_i(\gamma, \delta-1, \epsilon) + X_i(\gamma, \delta, \epsilon-1)$ equals

$$\binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i+1-\gamma-\delta-\epsilon} \binom{i}{\gamma, \zeta} \left\{ {i-\gamma-\zeta \atop \delta+\epsilon-1} \right\} (\alpha-\beta)^{\zeta}.$$

The Stirling numbers satisfy the recurrence

$$\left\{ {r \atop s} \right\} = s \left\{ {r-1 \atop s} \right\} + \left\{ {r-1 \atop s-1} \right\},$$

and therefore $(\delta+\epsilon)X_i(\gamma,\delta,\epsilon) + X_i(\gamma,\delta-1,\epsilon) + X_i(\gamma,\delta,\epsilon-1)$ equals

$$\binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i+1-\gamma-\delta-\epsilon} \binom{i}{\gamma,\zeta} \left\{ {i+1-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^{\zeta}.$$

The complete sum of five terms now reduces to

$$\binom{\delta+\epsilon}{\epsilon} \sum_{\zeta=0}^{i+1-\gamma-\delta-\epsilon} \binom{i+1}{\gamma,\zeta} \left\{ {i+1-\gamma-\zeta \atop \delta+\epsilon} \right\} (\alpha-\beta)^{\zeta} = X_{i+1}(\gamma,\delta,\epsilon),$$

and this completes the proof. □

5. The universal nonassociative enveloping algebra

Lemma 5.1. *The powers of $L(b)$ and $L(c)$ are*

$$L(b)^u = \sum_{\eta=0}^{u} \sum_{\theta=0}^{u-\eta} (-1)^{u-\eta-\theta} \binom{u}{\eta,\theta} S^{u-2\theta} M_b^{\eta} M_d^{u-\eta} D_c^{u-\eta},$$

$$L(c)^x = \sum_{\lambda=0}^{x} \sum_{\mu=0}^{x-\lambda} (-1)^{x-\lambda} \binom{x}{\lambda,\mu} S^{x-2\mu} M_c^{\lambda} M_d^{x-\lambda} D_b^{x-\lambda}.$$

Proof. We apply the trinomial theorem to the formulas for $L(b)$ and $L(c)$ in Table 2, since the terms in each operator commute:

$$L(b)^u = \sum_{\eta=0}^{u} \sum_{\theta=0}^{u-\eta} \binom{u}{\eta,\theta} (SM_b)^{\eta} \big(S^{-1}M_dD_c\big)^{\theta} \big(-SM_dD_c\big)^{u-\eta-\theta},$$

$$L(c)^x = \sum_{\lambda=0}^{x} \sum_{\mu=0}^{x-\lambda} \binom{x}{\lambda,\mu} \big(SM_c\big)^{\lambda} \big(-S^{-1}M_dD_b\big)^{\mu} \big(-SM_dD_b\big)^{x-\lambda-\mu}.$$

These formulas simplify as required using Lemma 3.2. □

Lemma 5.2. *The operator monomial of Lemma 4.3 equals*

$$L(a)^s S^t L(b)^u D_b^v D_c^w L(c)^x M_d^y L(d)^z = \sum_{\eta=0}^{u} \sum_{\theta=0}^{u-\eta} \sum_{\lambda=0}^{x} \sum_{\mu=0}^{x-\lambda} (-1)^{u-\eta-\theta+x-\lambda} \times$$

$$\binom{u}{\eta,\theta} \binom{x}{\lambda,\mu} M_a^s S^{t+u-2\theta+x-2\mu-z} M_b^{\eta} D_b^{v+x-\lambda} D_c^{u-\eta+w} M_c^{\lambda} M_d^{u-\eta+x-\lambda+y+z}.$$

Proof. Table 2 and Lemma 5.1 show that the operator monomial equals

$$\sum_{\eta=0}^{u}\sum_{\theta=0}^{u-\eta}\sum_{\lambda=0}^{x}\sum_{\mu=0}^{x-\lambda} M_a^s S^t (-1)^{u-\eta-\theta}\binom{u}{\eta,\theta} S^{u-2\theta} M_b^\eta M_d^{u-\eta} D_c^{u-\eta} D_b^v \times$$
$$D_c^w (-1)^{x-\lambda}\binom{x}{\lambda,\mu} S^{x-2\mu} M_c^\lambda M_d^{x-\lambda} D_b^{x-\lambda} M_d^y \left(S^{-1}M_d\right)^z,$$

which simplifies as required using Lemma 3.2. □

Lemma 5.3. *$L(a^i b^j c^k d^\ell)$ expands in terms of M_x, D_x and S to*

$$\sum_{\alpha=0}^{\min(j,k)}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{i}\sum_{\delta=0}^{i-\gamma}\sum_{\epsilon=0}^{i-\gamma-\delta}\sum_{\zeta=0}^{i-\gamma-\delta-\epsilon}\sum_{\eta=0}^{j-\alpha-\epsilon}\sum_{\theta=0}^{j-\alpha-\epsilon-\eta}\sum_{\lambda=0}^{k-\alpha-\delta}\sum_{\mu=0}^{k-\alpha-\delta-\lambda}$$
$$(-1)^{i+j+k+\alpha-\beta-\gamma-\epsilon-\eta-\theta-\lambda}\times$$
$$(\alpha-\beta)^\zeta \alpha!\binom{\alpha}{\beta}(\delta+\epsilon)!\binom{i}{\gamma,\zeta}\left\{\begin{matrix} i-\gamma-\zeta \\ \delta+\epsilon\end{matrix}\right\}\binom{j}{\alpha,\epsilon,\eta,\theta}\binom{k}{\alpha,\delta,\lambda,\mu}\times$$
$$M_a^\gamma S^{j+k-\ell-2\alpha-\beta-2\delta-2\epsilon-2\theta-2\mu} M_b^\eta D_b^{k-\alpha-\lambda} D_c^{j-\alpha-\eta} M_c^\lambda M_d^{j+k+\ell-\alpha-\eta-\lambda}.$$

Proof. In Lemma 5.2 we set $s=\gamma$, $t=-\beta-\delta-\epsilon$, $u=j-\alpha-\epsilon$, $v=\delta$, $w=\epsilon$, $x=k-\alpha-\delta$, $y=\alpha+\delta+\epsilon$, $z=\ell$ and obtain

$$L(a)^\gamma S^{-\beta-\delta-\epsilon} L(b)^{j-\alpha-\epsilon} D_b^\delta D_c^\epsilon L(c)^{k-\alpha-\delta} M_d^{\alpha+\delta+\epsilon} L(d)^\ell =$$
$$\sum_{\eta=0}^{j-\alpha-\epsilon}\sum_{\theta=0}^{j-\alpha-\epsilon-\eta}\sum_{\lambda=0}^{k-\alpha-\delta}\sum_{\mu=0}^{k-\alpha-\delta-\lambda}(-1)^{j-\epsilon-\eta-\theta+k-\delta-\lambda}\binom{j-\alpha-\epsilon}{\eta,\theta}\binom{k-\alpha-\delta}{\lambda,\mu}\times$$
$$M_a^\gamma S^{j+k-\ell-2\alpha-\beta-2\delta-2\epsilon-2\theta-2\mu} M_b^\eta D_b^{k-\alpha-\lambda} D_c^{j-\alpha-\eta} M_c^\lambda M_d^{j+k+\ell-\alpha-\eta-\lambda}.$$

We now combine this with Lemma 4.4 and Lemma 4.6. □

Definition 5.4. The **differential coefficients** are

$$\begin{bmatrix} r \\ 0\end{bmatrix}=1,\ \begin{bmatrix} r \\ s\end{bmatrix}=r(r-1)\cdots(r-s+1),\ \text{so that } D_x^s(x^r)=\begin{bmatrix} r \\ s\end{bmatrix}x^{r-s}.$$

In the next theorem we set $(\alpha-\beta)^\zeta=1$ when $\alpha=\beta$ and $\zeta=0$.

Theorem 5.5. *The product $(a^i b^j c^k d^\ell)(a^m b^n c^p d^q)$ in $U(\mathbb{M})$ equals*

$$\sum_{\alpha=0}^{\min(j,k)}\sum_{\beta=0}^{\alpha}\sum_{\gamma=0}^{i}\sum_{\delta=0}^{i-\gamma}\sum_{\epsilon=0}^{i-\gamma-\delta}\sum_{\zeta=0}^{i-\gamma-\delta-\epsilon}\sum_{\eta=0}^{j-\alpha-\epsilon}\sum_{\theta=0}^{j-\alpha-\epsilon-\eta}\sum_{\lambda=0}^{k-\alpha-\delta}\sum_{\mu=0}^{k-\alpha-\delta-\lambda}\sum_{\nu=0}^{m}$$
$$(-1)^{i+j+k+\alpha-\beta-\gamma-\epsilon-\eta-\theta-\lambda}(\alpha-\beta)^\zeta\alpha!\binom{\alpha}{\beta}(\delta+\epsilon)!\omega^\nu\times$$
$$\binom{i}{\gamma,\zeta}\left\{\begin{matrix} i-\gamma-\zeta \\ \delta+\epsilon\end{matrix}\right\}\binom{j}{\alpha,\epsilon,\eta,\theta}\binom{k}{\alpha,\delta,\lambda,\mu}\binom{m}{\nu}\begin{bmatrix} n \\ k-\alpha-\lambda\end{bmatrix}\begin{bmatrix} p+\lambda \\ j-\alpha-\eta\end{bmatrix}\times$$

$a^{m+\gamma-\nu} b^{-k+n+\alpha+\eta+\lambda} c^{-j+p+\alpha+\eta+\lambda} d^{j+k+\ell+q-\alpha-\eta-\lambda}$,

where $\omega = j+k-\ell-2\alpha-\beta-2\delta-2\epsilon-2\theta-2\mu$.

Proof. Apply the M_x, D_x, S operators in Lemma 5.3 to $a^m b^n c^p d^q$:

$$\begin{bmatrix} p+\lambda \\ j-\alpha-\eta \end{bmatrix} \begin{bmatrix} n \\ k-\alpha-\lambda \end{bmatrix} a^\gamma (a+\omega)^m b^{-k+n+\alpha+\eta+\lambda} c^{-j+p+\alpha+\eta+\lambda} d^{j+k+\ell+q-\alpha-\eta-\lambda}.$$

Use this in Lemma 5.3 and expand $(a+\omega)^m$. □

6. The universal alternative enveloping algebra

Definition 6.1. The **alternator ideal** in a nonassociative algebra A is generated by the elements (x,x,y) and (y,x,x) for all $x, y \in A$.

Definition 6.2. Let M be a Malcev algebra, $U(M)$ its universal enveloping algebra, and $I(M) \subseteq U(M)$ the alternator ideal. The **universal alternative enveloping algebra** of M is $A(M) = U(M)/I(M)$.

Lemma 6.3. *We have the following nonzero alternators in* $U(\mathbb{M})$*:*

$$(a, bc, bc) = 2d^2, \qquad (b, ac, ac) = cd, \qquad (c, ab, ab) = -bd.$$

Proof. Theorem 5.5 gives

$$\big(a(bc)\big)(bc) = ab^2c^2 - 2abcd + 2d^2, \qquad a\big((bc)(bc)\big) = ab^2c^2 - 2abcd,$$

which imply the first result. The other two are similar. □

Definition 6.4. Let $J \subseteq U(\mathbb{M})$ be the ideal generated by d^2, cd, bd. In $U(\mathbb{M})/J$ it suffices to consider two types of monomials, $a^i d$ and $a^i b^j c^k$, which we call type 1 and type 2 respectively. If m is one of these monomials, we write m when we mean $m + J$ in the next lemma.

Lemma 6.5. *In* $U(\mathbb{M})/J$ *we have*

(1) $(a^i d)(a^m d) = 0,$

(2) $(a^i b^j c^k)(a^m d) = \delta_{j0}\delta_{k0}\, a^{i+m} d,$

(3) $(a^i d)(a^m b^n c^p) = \delta_{n0}\delta_{p0}\, a^i (a-1)^m d,$

(4) $(a^i b^j c^k)(a^m b^n c^p) = a^i (a+j+k)^m b^{j+n} c^{k+p} + \delta_{j+n,1}\delta_{k+p,1} T^{im}_{jk},$

where

$$T^{im}_{jk} = \begin{cases} 0 & \text{if } (j,k) = (0,0), \\ (a-1)^{i+m} d - a^i (a+1)^m d & \text{if } (j,k) = (1,0), \\ -(a-1)^{i+m} d - a^i (a+1)^m d & \text{if } (j,k) = (0,1), \\ a^i (a-1)^m d - a^i (a+2)^m d & \text{if } (j,k) = (1,1). \end{cases}$$

Proof. We only need the terms in Theorem 5.5 in which the d-exponent is 0, or the d-exponent is 1 and the b- and c-exponents are 0.

For equation (1), we have $j = k = n = p = 0$, $\ell = q = 1$; hence $\min(j,k) = 0$, so $\alpha = 0$. The sums on η and λ are empty unless $\delta = 0$ and $\epsilon = 0$; hence $\eta = \lambda = 0$. Now each term in Theorem 5.5 has d-exponent $j{+}k{+}\ell{+}q{-}\alpha{-}\eta{-}\lambda = 2$; but $d^2 = 0$.

For equation (2), we have $\ell = n = p = 0$, $q = 1$. The d-exponent is $j{+}k{+}1{-}\alpha{-}\eta{-}\lambda$. This is 0 if and only if $\alpha{+}\eta{+}\lambda = j{+}k{+}1$; since $\alpha{+}\eta \le j$ and $\lambda \le k$ there are no solutions. The d-exponent is 1 if and only if $\alpha{+}\eta{+}\lambda = j{+}k$. Since $\eta \le j$, $\alpha{+}\eta \le j$, $\lambda \le k$, $\alpha{+}\lambda \le k$, the solution has $\eta = j$, $\lambda = k$. Therefore $\alpha = 0$, $\beta = 0$, and the sums on η, λ are empty unless $\delta = 0$, $\epsilon = 0$ so we get $\theta = \mu = 0$. We need $\zeta = 0$ to make the power of $\alpha{-}\beta$ nonzero. But $\zeta = i{-}\gamma$ since $\left\{ {r \atop 0} \right\} = \delta_{r0}$, and so $\gamma = i$. The sum collapses to

$$\sum_{\nu=0}^{m} (j{+}k)^{\nu} \binom{m}{\nu} a^{i+m-\nu} b^j c^k d = a^i (a{+}j{+}k)^m b^j c^k d.$$

Since $bd = cd = 0$, this is 0 unless $j = k = 0$.

For equation (3), we have $j = k = q = 0$, $\ell = 1$; hence $\min(j,k) = 0$, so $\alpha = \beta = 0$. The power of $\alpha{-}\beta$ is zero unless $\zeta = 0$. Since $j = \alpha = 0$, the sum on η is empty unless $\epsilon = 0$, so $\eta = \theta = 0$. Since $k = \alpha = 0$, the sum on λ is empty unless $\delta = 0$, so $\lambda = \mu = 0$. We are left with

$$\sum_{\gamma=0}^{i} \sum_{\nu=0}^{m} (-1)^{i-\gamma} (-1)^{\nu} \binom{i}{\gamma} \left\{ {i{-}\gamma \atop 0} \right\} \binom{m}{\nu} a^{m+\gamma-\nu} b^n c^p d.$$

The Stirling number is 0 unless $\gamma = i$, so we get

$$\delta_{n0}\delta_{p0} \sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu} a^{i+m-\nu} d = \delta_{n0}\delta_{p0}\, a^i (a{-}1)^m d,$$

since the monomial vanishes unless $n = p = 0$.

For equation (4), we have $\ell = q = 0$; the d-exponent is $j{+}k{-}\alpha{-}\eta{-}\lambda$. This is 0 if and only if $\alpha{+}\eta{+}\lambda = j{+}k$. As before $\eta = j$, $\lambda = k$; hence $\alpha = 0$, $\beta = 0$, and so $\delta = 0$, $\epsilon = 0$, $\theta = 0$, $\mu = 0$ and $\zeta = 0$. But $\zeta = i{-}\gamma$ since $\left\{ {r \atop 0} \right\} = \delta_{r0}$, and so $\gamma = i$. The sum collapses to

$$\sum_{\nu=0}^{m} (j{+}k)^{\nu} \binom{m}{\nu} a^{i+m-\nu} b^{j+n} c^{k+p} = a^i (a{+}j{+}k)^m b^{j+n} c^{k+p}.$$

If the d-exponent is 1, the b- and c-exponents are 0: $-k{+}n{+}\alpha{+}\eta{+}\lambda = 0$, $-j{+}p{+}\alpha{+}\eta{+}\lambda = 0$, $j{+}k{-}\alpha{-}\eta{-}\lambda = 1$. Adding the first and third (resp. second and third) gives $j{+}n = 1$ (resp. $k{+}p = 1$), so we have four cases: $(a^i)(a^m bc)$, $(a^i b)(a^m c)$, $(a^i c)(a^m b)$, $(a^i bc)(a^m)$.

Case 1: $jknp = 0011$. We have $(a^i)(a^m bc) = a^{i+m}bc$, so there is no term with d-exponent 1.

Case 2: $jknp = 1001$. We have $\alpha = \beta = 0$ and hence $\zeta = 0$. The λ-sum is empty unless $\delta = 0$, and then $\lambda = \mu = 0$. The η-sum is empty unless $\epsilon \in \{0,1\}$, so we have four subcases: $(\epsilon,\eta,\theta) = (0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$; the last case occurs only when $\gamma < i$. For $(0,0,0)$ the exponent of -1 is $i{+}1{-}\gamma$; otherwise it is $i{-}\gamma$. For $(0,0,0)$, $(0,1,0)$ the factor ω^ν is 1; otherwise it is $(-1)^\nu$. If $\gamma < i$ then the Stirling number is $\delta_{\epsilon 1}$ (so $\eta = \theta = 0$); otherwise it is $\delta_{\epsilon 0}$. The monomial for $(0,1,0)$ when $\gamma = i$ has d-exponent 0, contradicting our assumption, so this term does not appear. The sum collapses to

$$\sum_{\gamma=0}^{i}\sum_{\nu=0}^{m}(-1)^{i-\gamma}(-1)^{\nu}\binom{i}{\gamma}\binom{m}{\nu}a^{\gamma+m-\nu}d - \sum_{\nu=0}^{m}\binom{m}{\nu}a^{i+m-\nu}d,$$

which gives the result.

Case 3: $jknp = 0110$. Similar to Case 2.

Case 4: $jknp = 1100$. We have $\alpha \in \{0,1\}$. There are three cases: $(\alpha,\beta) = (0,0)$, $(1,0)$, $(1,1)$. The d-exponent is $2{-}\alpha{-}\eta{-}\lambda$; by assumption this is 1, so $\alpha{+}\eta{+}\lambda = 1$. For $(\alpha,\beta) = (1,1)$ we must have $\delta = 0$ and then $\lambda = \mu = 0$; likewise $\epsilon = 0$ and then $\eta = \theta = 0$. Furthermore $\zeta = 0$ and $\gamma = i$. The sum collapses to

$$\sum_{\nu=0}^{m}(-1)^{\nu}\binom{m}{\nu}a^{i+m-\nu}d = a^i(a{-}1)^m d.$$

For $(\alpha,\beta) = (1,0)$ the sum collapses to

$$-\sum_{\gamma=0}^{i}(-1)^{i-\gamma}\binom{i}{\gamma}a^{\gamma+m}d = -(a{-}1)^i a^m d.$$

For $(\alpha,\beta) = (0,0)$ the sum collapses to

$$\sum_{\gamma=0}^{i}\sum_{\epsilon=0}^{i-\gamma}\sum_{\theta=0}^{1-\epsilon}\sum_{\nu=0}^{m}(-1)^{i-\gamma-\epsilon-\theta-1}(2{-}2\epsilon{-}2\theta)^{\nu}\binom{i}{\gamma}\left\{ {i-\gamma \atop \epsilon} \right\}\binom{m}{\nu}a^{\gamma-\nu+m}d.$$

The sum on θ gives $\epsilon \in \{0,1\}$. If $\gamma < i$ then $\epsilon = 1$; hence $\theta = 0$ and $\nu = 0$. If $\gamma = i$ then $\epsilon = 0$. We separate the last term of the γ-sum:

$$\sum_{\gamma=0}^{i-1}(-1)^{i-\gamma}\binom{i}{\gamma}a^{\gamma+m}d + \left[\sum_{\theta=0}^{1}\sum_{\nu=0}^{m}(-1)^{-\theta-1}(2{-}2\theta)^{\nu}\binom{m}{\nu}a^{i-\nu+m}d\right].$$

The first term cancels with the result for $(\alpha,\beta) = (1,0)$. □

The following theorem was first established by Shestakov using different methods; a similar result appears in [8, Example 1].

Theorem 6.6. *The universal alternative enveloping algebra* $A(\mathbb{M})$ *is isomorphic to the algebra with basis* $\{ a^i d,\ a^i b^j c^k \mid i,j,k \geq 0 \}$ *and structure constants of Lemma 6.5.*

Proof. Once we show that $U(\mathbb{M})/J$ is alternative, it follows that J equals the alternator ideal $I(M)$ and hence that $U(\mathbb{M})/J$ is isomorphic to $A(\mathbb{M})$. We prove alternativity by showing that the associator alternates. Since the associator is multilinear, it suffices to consider monomials. We use Lemma 6.5 repeatedly. Since the product of a monomial of type 1 with any monomial is a linear combination of monomials of type 1, every associator with two monomials of type 1 vanishes. We next consider one monomial of type 1 and two of type 2. Since the T-term in Equation (4) contains only monomials of type 1, $(a^i d, a^m b^n c^p, a^r b^s c^t)$ equals

$$\begin{aligned}&\big[\delta_{n0}\delta_{p0}a^i(a{-}1)^m d\big](a^r b^s c^t) - (a^i d)\big[a^m(a{+}n{+}p)^r b^{n+s}c^{p+t} + T^{**}_{**}\big] = \\ &\delta_{n0}\delta_{p0}\delta_{s0}\delta_{t0}a^i(a{-}1)^{m+r}d - \delta_{n+s,0}\delta_{p+t,0}a^i(a{-}1)^m(a{-}1{+}n{+}p)^r d = 0.\end{aligned}$$

Similarly $(a^i b^j c^k, a^m d, a^r b^s c^t) = (a^i b^j c^k, a^m b^n c^p, a^r d) = 0$. We finally consider three monomials of type 2: $(a^i b^j c^k, a^m b^n c^p, a^r b^s c^t)$ equals

$$\begin{aligned}&\big[a^i(a{+}j{+}k)^m b^{j+n}c^{k+p} + \delta_{j+n,1}\delta_{k+p,1}T^{im}_{jk}\big](a^r b^s c^t) \\ &\quad - (a^i b^j c^k)\big[a^m(a{+}n{+}p)^r b^{n+s}c^{p+t} + \delta_{n+s,1}\delta_{p+t,1}T^{mr}_{np}\big].\end{aligned}$$

We write this as $A - B + C - D$ where

$$\begin{aligned}A &= \big[a^i(a{+}j{+}k)^m b^{j+n}c^{k+p}\big](a^r b^s c^t),\\ B &= (a^i b^j c^k)\big[a^m(a{+}n{+}p)^r b^{n+s}c^{p+t}\big],\\ C &= \delta_{j+n,1}\delta_{k+p,1}T^{im}_{jk}(a^r b^s c^t), \qquad D = \delta_{n+s,1}\delta_{p+t,1}(a^i b^j c^k)T^{mr}_{np}.\end{aligned}$$

Expanding $(a{+}j{+}k)^m$ and $(a{+}n{+}p)^r$ we see that $A - B$ equals

$$\delta_{j+n+s,1}\delta_{k+p+t,1}\left[\sum_{\nu=0}^{m}\binom{m}{\nu}(j{+}k)^\nu T^{i+m-\nu,r}_{j+n,k+p} - \sum_{\xi=0}^{r}\binom{r}{\xi}(n{+}p)^\xi T^{i,m+r-\xi}_{jk}\right].$$

For $jknpst = 110000$ we get

$$\begin{aligned}A - B &= \sum_{\nu=0}^{m}\binom{m}{\nu}2^\nu T^{i+m-\nu,r}_{11} - T^{i,m+r}_{11}\\ &= a^i(a{+}2)^m(a{-}1)^r d - a^i(a{+}2)^{m+r}d - a^i(a{-}1)^{m+r}d + a^i(a{+}2)^{m+r}d\\ &= a^i(a{-}1)^r(a{+}2)^m d - a^i(a{-}1)^{m+r}d.\end{aligned}$$

Similar calculations give

$$
\begin{aligned}
jknpst = 100100&: \quad A - B = a^i(a{-}1)^r(a{+}1)^m d - a^r(a{-}1)^{i+m} d,\\
jknpst = 100001&: \quad A - B = a^m(a{-}1)^{i+r} d - (a{-}1)^{i+m+r} d,\\
jknpst = 011000&: \quad A - B = a^i(a{-}1)^r(a{+}1)^m d + (a{-}1)^{i+m} a^r d,\\
jknpst = 001100&: \quad A - B = a^{i+m}(a{-}1)^r d - a^{i+m}(a{+}2)^r d,\\
jknpst = 001001&: \quad A - B = (a{-}1)^{i+m+r} d - a^{i+m}(a{+}1)^r d,\\
jknpst = 010010&: \quad A - B = -a^m(a{-}1)^{i+r} d + (a{-}1)^{i+m+r} d,\\
jknpst = 000110&: \quad A - B = -(a{-}1)^{i+m+r} d - a^{i+m}(a{+}1)^r d,\\
jknpst = 000011&: \quad A - B = 0.
\end{aligned}
$$

For C and D we obtain

$$
\begin{aligned}
jknp = 1100&: \quad C = \delta_{s0}\delta_{t0} a^i(a{-}1)^{m+r} d - \delta_{s0}\delta_{t0} a^i(a{-}1)^r(a{+}2)^m d,\\
jknp = 1001&: \quad C = \delta_{s0}\delta_{t0}(a{-}1)^{i+m+r} d - \delta_{s0}\delta_{t0} a^i(a{-}1)^r(a{+}1)^m d,\\
jknp = 0110&: \quad C = -\delta_{s0}\delta_{t0}(a{-}1)^{i+m+r} d - \delta_{s0}\delta_{t0} a^i(a{-}1)^r(a{+}1)^m d,\\
jknp = 0011&: \quad C = 0,\\
npst = 1100&: \quad D = \delta_{j0}\delta_{k0} a^{i+m}(a{-}1)^r d - \delta_{j0}\delta_{k0} a^{i+m}(a{+}2)^r d,\\
npst = 1001&: \quad D = \delta_{j0}\delta_{k0} a^i(a{-}1)^{m+r} d - \delta_{j0}\delta_{k0} a^{i+m}(a{+}1)^r d,\\
npst = 0110&: \quad D = -\delta_{j0}\delta_{k0} a^i(a{-}1)^{m+r} d - \delta_{j0}\delta_{k0} a^{i+m}(a{+}1)^r d,\\
npst = 0011&: \quad D = 0.
\end{aligned}
$$

We combine these results to get $A - B + C - D$:

$$
\begin{aligned}
jknpst = 110000&: \quad (a^i bc, a^m, a^r) = 0,\\
jknpst = 100100&: \quad (a^i b, a^m c, a^r) = (a{-}1)^{i+m+r} d - a^r(a{-}1)^{i+m} d,\\
jknpst = 100001&: \quad (a^i b, a^m, a^r c) = a^m(a{-}1)^{i+r} d - (a{-}1)^{i+m+r} d,\\
jknpst = 011000&: \quad (a^i c, a^m b, a^r) = -(a{-}1)^{i+m+r} d + (a{-}1)^{i+m} a^r d,\\
jknpst = 001100&: \quad (a^i, a^m bc, a^r) = 0,\\
jknpst = 001001&: \quad (a^i, a^m b, a^r c) = (a{-}1)^{i+m+r} d - a^i(a{-}1)^{m+r} d,\\
jknpst = 010010&: \quad (a^i c, a^m, a^r b) = -a^m(a{-}1)^{i+r} d + (a{-}1)^{i+m+r} d,\\
jknpst = 000110&: \quad (a^i, a^m c, a^r b) = -(a{-}1)^{i+m+r} d + a^i(a{-}1)^{m+r} d,\\
jknpst = 000011&: \quad (a^i, a^m, a^r bc) = 0.
\end{aligned}
$$

The alternativity property is now clear. □

7. Conclusion

Since the alternator ideal $I(\mathbb{M})$ contains no elements of degree 1, the natural mapping from $\mathbb{M}$ to $A(\mathbb{M})$ is injective, and hence $\mathbb{M}$ is special. This also follows directly from the isomorphism $\mathbb{M} \cong \mathbb{A}^-$ where $\mathbb{A}$ is the algebra in Table 3. For any $x, y, z \in \mathbb{A}$ we write $x = (x_1, \ldots, x_4)$ etc. and calculate the associator to prove that $\mathbb{A}$ is alternative:

$$(xy)z - x(yz) = \left[0, 0, 0, -\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \right].$$

This is analogous to the construction of the split simple Lie algebra $sl_2(\mathbb{F})$ as a subalgebra (the trace-zero matrices) of the commutator algebra of the associative algebra $M_2(\mathbb{F})$ of 2×2 matrices over $\mathbb{F}$.

Table 3. The 4-dimensional alternative algebra $\mathbb{A}$

$\cdot$	a	b	c	d
a	a	0	0	d
b	b	0	d	0
c	c	$-d$	0	0
d	0	0	0	0

8. Acknowledgements

We thank J. M. Pérez-Izquierdo and I. P. Shestakov for helpful comments; in particular, Shestakov sent us his structure constants for the universal alternative enveloping algebra $A(\mathbb{M})$. We thank Antonio Behn for telling us about the alternative algebra $\mathbb{A}$ of Table 3. Bremner, Hentzel and Peresi thank BIRS for its hospitality during our Research in Teams program in May 2005. Bremner and Usefi were partially supported by NSERC. Peresi was partially supported by CNPq.

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Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada
E-mail address: bremner@math.usask.ca

Department of Mathematics, Iowa State University, Ames, U.S.A.
E-mail address: hentzel@iastate.edu

Departamento de Matemática, Universidade de São Paulo, Brasil
E-mail address: peresi@ime.usp.br

Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada. *Current address:* Department of Mathematics, University of British Columbia, Vancouver, Canada
E-mail address: usefi@math.ubc.ca