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Restricted Lie algebras with subexponential growth

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Dedicated to Professor Donald S. Passman on the occasion of his 65^{th} *birthday.*

ABSTRACT. Let L be a restricted Lie algebra generated by a finite set X. We define its growth $g(n)$ to be $p^{\dim W_n}$, where W_n is the space spanned by all restricted monomials in X of degree at most n. First we deduce from a theorem of Passman and Petrogradsky that $g(n)$ grows polynomially if and only if L is virtually nilpotent. Subsequently, we prove that if $g(n)$ grows subexponentially then the lower central series of L must stabilize. This yields a gap between polynomial and exponential growth in the class of all residually nilpotent restricted Lie algebras.

1. Introduction

There is a long history of studying groups and algebras in terms of their growth functions. A prime result is Gromov's celebrated theorem ([**Gro**]), which states that a finitely generated group has polynomial growth if and only if it is virtually nilpotent. Grigorchuk showed in ([**Gri3**]) that the polynomial growth condition in Gromov's theorem can be weakened to subradical growth provided the group is known to be residually a finite p-group. Quite recently, Wilson ([**Wil**]) generalized Grigorchuk's theorem to the class of all residually soluble groups. Grigorchuk ([**Gri1**], [**Gri2**]) also constructed the first known finitely generated groups of intermediate growth; that is, groups having growth that lies strictly between polynomial and exponential. Associative algebras with polynomial growth are less well understood, but it is known ([**KL**]) that there exist finitely generated associative algebras with intermediate growth.

In this paper, we examine growth functions in the category of restricted Lie algebras.

For any positive real-valued sequences f, g, we write $f \preceq g$ if there exists an integer constant C such that $f(n) \leq C + Cg(Cn + C)$ for all n, and write $f \sim g$ if both $f \preceq g$ and $g \preceq f$. If there exists a positive integer d such that $g(n) \preceq n^d$ then

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we say g has polynomial growth, if $g(n) \prec e^n$ then we say g has subexponential growth, and if $g(n) \sim e^n$ then we say g has exponential growth. Finally, if g has subexponential growth but does not have polynomial growth then g is said to have intermediate growth.

Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic p. Suppose further that L is generated by a finite set X. We define

$$W(X,n) = \langle [x_1, \dots, x_i]^{p^j} \mid x_1, \dots, x_i \in X, ip^j \leq n \rangle_{\mathbb{F}}.$$

Here the associated growth function g of L with respect to X is defined by $g(n) = p^{\dim_{\mathbb{F}} W(X,n)}$, rather than the more natural $\dim_{\mathbb{F}} W(X,n)$. This scaling is made to facilitate an analogy between restricted Lie algebras and groups with the same type of growth. It is easy to see that if g' is the growth function corresponding to another generating set X' of L then $g \sim g'$. We shall say that L has polynomial, subexponential, *etc*, growth when its corresponding growth function has the respective properties.

Recently, Passman and Petrogradsky ([**PP**]) gave several characterizations of when a finitely generated restricted Lie algebra has polynomial growth (that is, $L_p \dim L < \infty$ in their notation). We shall deduce from their results that the exact restricted-Lie analogue of Gromov's theorem holds; namely, a finitely generated restricted Lie algebra has polynomial growth precisely when it is virtually nilpotent. Our primary goal, however, is to demonstrate that the lower central series of every finitely generated restricted Lie algebra with subexponential growth stabilizes. Consequently, there are no residually nilpotent restricted Lie algebras of intermediate growth.

2. Preliminaries

Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic p. As is customary, we shall denote the product and p-map in L by $(x,y) \mapsto [x,y]$ and $x \mapsto x^p$, respectively. The lower central series of L is defined recursively by $\gamma_1(L) = L$ and $\gamma_n(L) = [\gamma_{n-1}(L), L]$, for every $n \geq 2$. Recall that L is said to be nilpotent if $\gamma_{n+1}(L) = 0$ for some n, the least such n being the class of L. Also recall that L is called residually nilpotent if $\cap_{n\geq 1}\gamma_n(L) = 0$ and that L is said to be virtually nilpotent if it contains a nilpotent ideal of finite codimension in L.

For any subset S of L and positive integer j, we denote by S^{p^j} the $\mathbb{F}$-subspace of L spanned by the elements x^{p^j} with $x \in S$. We shall denote by $\langle S \rangle$ the Lie subalgebra generated by S and by $\langle S \rangle_p$ the restricted Lie subalgebra generated by S. For each integer $n \geq 1$, we define

$$D_n(L) = \sum_{ip^j \geq n} \gamma_i(L)^{p^j}.$$

The restricted ideal $D_n(L)$ is sometimes called the n^{th} dimension subalgebra of L and arises naturally in the context of the restricted universal enveloping algebra of L (see [**RS**]). It follows from the defining axioms of a restricted Lie algebra that for all x, y in L we have $[x, y^p] = x(\operatorname{ad} y)^p$ and $(x+y)^p = x^p + y^p$ modulo $\gamma_p(\langle x, y \rangle)$, and consequently that $D_m(L)^p \subseteq D_{pm}(L)$ and $[D_m(L), D_n(L)] = \gamma_{m+n}(L)$ for every m, n. We shall use these basic facts without explicit reference. For each integer $n \geq 1$, we put $d_n = \dim_{\mathbb{F}} D_n(L)/D_{n+1}(L)$ and $a_n = \dim_{\mathbb{F}} \gamma_n(L) + D_{n+1}(L)/D_{n+1}(L)$. We also set $\rho_n = p^{\dim_{\mathbb{F}} L/D_n(L)}$. It is clear that $\rho_{n+1} \leq g(n)$, for every $n \geq 1$.

We record now some simple facts for later use.

LEMMA 2.1. *The following statements hold in every restricted Lie algebra L.*
(1) *If m is a positive integer such that $p \nmid m$ then $d_m = a_m$.*
(2) *If n is such that $d_n = 0$ then $\gamma_{n+1}(L) = \gamma_{n+2}(L)$.*
(3) *If the lower central series of L stabilizes then $d_n(L) = 0$, for infinitely many n.*

PROOF. Notice that if $p \nmid m$ and $i < m$ then $ip^j \geq m$ implies $ip^j \geq m+1$. It follows that if $p \nmid m$ then $d_m = a_m$. Suppose next that n is such that $d_n = 0$. Then $\gamma_{n+1}(L) = [D_n(L), L] = [D_{n+1}(L), L] = \gamma_{n+2}(L)$. For the final statement suppose that $\gamma_m(L) = \gamma_{m+1}(L)$. Then clearly $a_n = 0$ for every $n \geq m$. However, by the first statement, if $p \nmid n$ we have $d_n = a_n$. Consequently, $d_n = 0$ for every $n \geq m$ that is relatively prime to p. □

3. Polynomial growth

We are ready to deduce from Passman and Petrogradsky's main result in [**PP**] the exact analogue of Gromov's theorem. We require this new form in the proof of our main result, Theorem C.

THEOREM A. *Suppose that L is a restricted Lie algebra over a field of characteristic p. The following properties are equivalent.*
(1) *L is finitely generated and has polynomial growth.*
(2) *L is poly-CF; that is, L has a finite series whose factors are either cyclic or finite-dimensional.*
(3) *$L = H_p$, where H is a finite-dimensional Lie subalgebra of L.*
(4) *$L = R \oplus Z$, where R is a finite-dimensional Lie subalgebra and Z is a central free abelian restricted Lie subalgebra of finite rank.*
(5) *L is virtually nilpotent.*

PROOF. The equivalence of (1)-(4) comes directly from [**PP**]. To see why (5) implies (2), suppose that L has a nilpotent ideal J of finite codimension in L. Without loss, we may assume that J is restricted. Let c be the class of J. Since L is finitely generated, it follows from [**BKS**] that J is also finitely generated (when viewed as a restricted Lie algebra). This forces the finite-dimensionality of $J/D_c(J)$. Next observe that $[J, D_c(J)] \subseteq \gamma_{c+1}(J) = 0$. Consequently, $D_c(J)$ is an abelian restricted subalgebra of finite codimension in L. Since any restricted subalgebra of finite codimension contains a restricted ideal of finite codimension (see [**BMPZ**]), it follows that L contains an abelian restricted ideal A of finite codimension. But then A is generated as a restricted Lie algebra by a finite set X. Therefore $A = H_p$, where $H = \langle X \rangle_F$. The implication (3)⇒(2) applied to A now yields that A is poly-CF. But then L is poly-CF, too, as required. Finally, to prove (4)⇒(5) we need only point out that Z contains a restricted ideal of finite codimension in L. □

4. Uniserial actions

Our main technique is an application of the theory of uniserial modules developed in [**RSe**] to solve the so-called Coclass Conjectures in the category of restricted Lie algebras.

Recall that L is said to be p-nilpotent if $L^{p^k} = 0$ for some k. We say that L lies in the class $\mathcal{F}_p$ if it is both finite-dimensional and p-nilpotent. It follows from Engel's theorem that such an L is nilpotent. An L-module V is a vector space over $\mathbb{F}$ endowed with a bilinear map $[\ ,\ ] : V \times L \to V$ satisfying $[v,[x,y]] = [[v,x],y] - [[v,y],x]$ and $[v,x^p] = [v,_p x]$, for all $x,y \in L$ and $v \in V$.

For a subspace W of V, we shall denote by $[W,L]_L$ the L-submodule of V generated by all the elements of the form $[w,x]$, where $w \in W$ and $x \in L$. A section of V is a quotient of L-submodules of V. An L-module V is said to be uniserial if every section of V on which L acts trivially has dimension at most 1. We let $V_0 = V$ and recursively define $V_i = [V_{i-1},L]_L$ for every $i \geq 1$.

Lemma 4.1. *Suppose that $L \in \mathcal{F}_p$ and let V be an L-module with $dim_{\mathbb{F}}(V) = n$. Then V is uniserial if and only if $V_{n-1} \neq 0$.*

Proof. Suppose that $V_{n-1} \neq 0$. Then, since L is nilpotent, we have the following strictly descending series of L-submodules

$$V = V_0 > V_1 > \cdots > V_{n-1} > V_n = 0,$$

where each factor has dimension 1. In order to prove that V is uniserial, it is enough to show that the V_i are the only submodules of V. We prove this by induction on n, the case $n = 1$ being trivial. Suppose then that $n \geq 2$ and that the assertion is true for all modules of dimension at most $n-1$. Now if W is a proper submodule of V then $W \cap V_1 = V_i$ for some $i \geq 1$, by the induction hypothesis applied to V_1. If $W \leq V_1$ then $W = V_i$. Otherwise, we must have $W + V_1 = V$, $[W,L]_L \leq W \cap V_1 = V_i$, and $i \geq 2$. However, then $V_1 = [W+V_1,L]_L = [W,L]_L + [V_1,L]_L = V_2$, a contradiction. The converse is trivial. $\square$

Next we recall the following key lemma from [**RSe**].

Lemma 4.2. *Suppose that $L \in \mathcal{F}_p$ acts uniserially on a module W. Then $[W_j, D_{p^i}(L)]_L = W_{j+p^i}$ for all $i,j \geq 0$.*

Lemma 4.3. *Let L be a restricted Lie algebra such that d_1 is finite and suppose that $d_i = 1$, for every $p^j \leq i \leq 2p^j$, where $j \geq 1$. Then $a_{2p^j} = 0$ and so $d_{2p^j+1} = 0$.*

Proof. We consider first the case when $L \in \mathcal{F}_p$. Regard the section $W = D_{p^j}/D_{2p^j+1}$ as an L-module. We claim that W cannot be uniserial. Indeed, suppose otherwise. Then since $\dim_{\mathbb{F}} W = p^j + 1$, $W_{p^j} \neq 0$ by Lemma 4.1. However, because $D_{p^j}(L)/D_{p^j+1}(L)$ is one-dimensional, clearly we have $W_{p^j} = [W, D_{p^j}(L)] = 0$, by Lemma 4.2. Hence, W is not uniserial; in other words, $W_{p^j} = 0$ by Lemma 4.1. It follows immediately that $a_{2p^j} = 0$. Hence, by Lemma 2.1, we have $d_{2p^j+1}(L) = 0$. To extend the result to an arbitrary restricted Lie algebra L, let $\bar{L} = L/D_{2p^j+2}(L)$. Once we observe that $\bar{L} \in \mathcal{F}_p$, $a_i(\bar{L}) = a_i(L)$ and $d_i(\bar{L}) = d_i(L)$, for every $1 \leq i \leq 2p^j + 1$, the proof is complete. $\square$

4. Subexponential growth

We are now ready to prove our main results. We begin with a more precise technical result. For a real number r we denote by $\lfloor r \rfloor$ the greatest integer less than or equal to r.

THEOREM B. *Let L be a restricted Lie algebra over a field of characteristic p.*

(1) *If p is odd and $m \geq 2$ is such that $\rho_m < p^{m+\lfloor \log_p m \rfloor - 1}$, then $d_n = 0$ for some $n \leq m-1$. In particular, $\gamma_m(L) = \gamma_{m+1}(L)$.*

(2) *Fix an integer $m \geq 4$ and let k be the unique odd integer such that $2^k \leq m < 2^{k+2}$. If $p = 2$ and $\rho_m < 2^{m+(k-1)/2}$, then $d_n = 0$ for some $n \leq m-1$. In particular, $\gamma_m(L) = \gamma_{m+1}(L)$.*

PROOF OF (1). Suppose, to the contrary, that $\rho_m < p^{m+\lfloor \log_p m \rfloor - 1}$ and yet $d_n \geq 1$ for every $n \leq m-1$. If $m < p$ then, by hypothesis, $d_1 + \cdots + d_{m-1} < m-1$ and so $d_n = 0$ for some $n \leq m-1$. Thus, we may safely assume that $m \geq p$; in other words, $k = \lfloor \log_p m \rfloor \geq 1$. Next notice that $\rho_{p^k} < p^{p^k+k-1}$ for otherwise

$$\begin{aligned}\rho_m &= \rho_{p^k} p^{d_{p^k} + \cdots + d_{m-1}} \\ &\geq p^{p^k+k-1} p^{m-p^k} = p^{m+k-1} = p^{m+\lfloor \log_p m \rfloor - 1},\end{aligned}$$

a contradiction. Consequently, we need only to consider the case when $m = p^k$ and $k \geq 1$. We claim next that $d_1 \geq 2$. Indeed, otherwise $d_1 = 1$ and so $\gamma_2(L) = [L, D_2(L)] = \gamma_3(L)$. However, since p is odd, this would imply that $d_2 = a_2 = 0$ by Lemma 2.1, a contradiction. Thus, $d_1 \geq 2$ as claimed. Also, again using the fact that p is odd, we have $p^j \leq 2p^j \leq p^{j+1} - 1$ for every $j \geq 1$. Therefore, according to Lemma 4.3, for every $j \leq k-1$ there exists an integer r_j in the range $p^j \leq r_j \leq p^{j+1} - 1$ such that $d_{r_j} \geq 2$. This now yields

$$\begin{aligned}\rho_{p^k} &= p^{(d_1+\cdots+d_{p-1})+(d_p+\cdots+d_{p^2-1})+\cdots+(d_{p^{k-1}}+\cdots+d_{p^k-1})} \\ &\geq p^{p+(p^2-p+1)+\cdots+(p^k-p^{k-1}+1)} = p^{p^k+k-1},\end{aligned}$$

a contradiction.

PROOF OF (2). Suppose we have $\rho_m < 2^{m+(k-1)/2}$ and yet $d_n \geq 1$ for all $n \leq m-1$. If $d_1 = 1$ then $\gamma_2(L) = \gamma_3(L)$, and so $d_3 = a_3 = 0$ by Lemma 2.1, contradicting our assumption that $m \geq 4$. Thus, $d_1 \geq 2$. Now observe that if $m < 8$ then $d_1 + \cdots + d_{m-1} < m$ and so $d_n = 0$ for some $n \leq m-1$, a contradiction. Thus, $m \geq 8$; that is, $k \geq 3$. Next notice that $\rho_{2^k} < 2^{2^k+(k-1)/2}$ for otherwise

$$\begin{aligned}\rho_m &= \rho_{2^k} 2^{d_{2^k}+\cdots+d_{m-1}} \\ &\geq 2^{2^k+(k-1)/2} 2^{m-2^k} = 2^{m+(k-1)/2};\end{aligned}$$

therefore, it suffices to consider only the case when $m = 2^k$ and $k \geq 3$. From Lemma 4.3 it follows that, for every odd integer $j \leq k-2$, there exists an integer r_j in the range $2^j \leq r_j \leq 2^{j+2} - 1$ such that $d_{r_j} \geq 2$. Now we have

$$\begin{aligned}\rho_{2^k} &= 2^{d_1+(d_2+\cdots+d_{2^3-1})+\cdots+(d_{2^{k-2}}+\cdots+d_{2^k-1})} \\ &\geq 2^{2+(2^3-2+1)+\cdots+(2^k-2^{k-2}+1)} \\ &= 2^{2^k+(k-1)/2},\end{aligned}$$

our final contradiction. □

Consider the following consequence of the odd characteristic case of Theorem B: if $\rho_{p^k} < p^{p^k+k-1}$ for some k then $d_n = 0$ for some n. This bound is best possible. Indeed, consider the 1-generator restricted Lie algebras $H = \langle x | \, x^p = 0 \rangle$ and $K = \langle y \rangle$. One can present their wreath product $L = H \wr K$ by

$$L = \langle x, y | \, a_i = x(\operatorname{ad} y)^i, [a_i, a_j] = 0, a_i^p = 0, i, j \geq 0 \rangle.$$

It is not difficult to check that $d_n = 2$ for each n a power of p and $d_n = 1$ otherwise. This leads easily to the fact that $\rho_{p^k} = p^{p^k+k-1}$ for all k.

We are finally ready to state our main result.

THEOREM C. *Let L be a finitely generated residually nilpotent restricted Lie algebra. The following statements are equivalent.*

(1) *L has subexponential growth.*
(2) *$d_n = 0$, for some n.*
(3) *L is nilpotent.*
(4) *L has polynomial growth.*

PROOF. The implication (1)⇒(2) follows immediately from Theorem B and the fact that $\rho_{n+1} \leq g(n)$ for any growth function g of L. Lemma 2.1 shows (2)⇒(3). Theorem A yields (3)⇒(4). The last implication, (4)⇒(1), is trivial. □

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