THE ISOMORPHISM PROBLEM FOR UNIVERSAL ENVELOPING ALGEBRAS OF LIE ALGEBRAS

DAVID RILEY AND HAMID USEFI

Abstract. Let L be a Lie algebra with universal enveloping algebra $U(L)$. We prove that if H is another Lie algebra with the property that $U(L) \cong U(H)$ then certain invariants of L are inherited by H. For example, we prove that if L is nilpotent then H is nilpotent with the same class as L. We also prove that if L is nilpotent of class at most two then L is isomorphic to H.

1. Introduction

We shall say that a particular invariant of a Lie algebra L is *determined* by its (universal) enveloping algebra, $U(L)$, if every Lie algebra H also possesses this invariant whenever $U(L)$ and $U(H)$ are isomorphic as associative algebras. Thus, roughly speaking, an invariant of L is determined by $U(L)$ whenever it can be deduced from the algebraic structure of $U(L)$ without any direct knowledge of the underlying Lie algebra L itself. For example, it is well-known that the dimension of a finite-dimensional Lie algebra L is determined by $U(L)$ since it coincides with the Gelfand-Kirillov dimension of $U(L)$. The main purpose of this paper is to demonstrate that certain other invariants of L are also determined by $U(L)$.

The most far reaching problem of this sort is the isomorphism problem for enveloping algebras. It asks whether or not (the isomorphism type of) every Lie algebra L is determined by $U(L)$. This problem has its historical roots in the corresponding isomorphism problem for group rings: is every finite group G determined by (the ring-theoretic properties of) its integral group ring, $\mathbb{Z}G$? A positive solution for the class of all nilpotent groups was given independently in [12] and [15]. The isomorphism problem for group rings does, however, have a negative solution in general (see [5]). We remark that easier counterexamples exist when the base ring $\mathbb{Z}$ is replaced by a field.

Although the isomorphism problem for enveloping algebras is well-known among researchers, few results have appeared in the literature. An inspection of an example constructed by Mikhalev, Umbirbaev and Zolotykh for

2000 *Mathematics Subject Classification.* Primary 17B35; Secondary 16R10, 17B01, 20C05.

The research of the first author is supported by NSERC of Canada.

another purpose (see Theorem 28.10 in [9]) readily provides a counterexample to the isomorphism problem for enveloping algebras of Lie algebras when posed in its most general form.

Example A. *Let $\mathbb{F}$ be a field of odd characteristic p and let $L(X)$ be the free Lie algebra generated by $X = \{x, y, z\}$ over $\mathbb{F}$. Set $h = x + [y, z] + (ad\ x)^p(z) \in L(X)$ and put $L = L(X)/\langle h\rangle$, where $\langle h\rangle$ denotes the ideal generated by h in $L(X)$. Then L is not a free Lie algebra even though $U(L)$ is freely generated on 2 generators (and so $U(L)$ is isomorphic to the universal enveloping algebra of the 2-generator free Lie algebra).*

An interesting open problem asks whether or not similar examples can occur in characteristic zero; that is, does there exist a non-free Lie algebra L over a field of characteristic zero such that $U(L)$ is a free associative algebra?

Given the existence of Example A, it makes sense to focus the isomorphism problem on specific classes of Lie algebras. As mentioned above, it is known that the dimension of a finite-dimensional Lie algebra is determined. Thus, if L is also abelian then L is determined by $U(L)$. There is also a recent low-dimensional result. Based on results for simple Lie algebras in [8], it was shown in [3] that L is determined by $U(L)$ in the case when L is any Lie algebra of dimension at most three over a field of any characteristic other than two.

Motivated by the positive group-ring-theoretic results, it seems natural to direct our attention to the isomorphism problem for nilpotent Lie algebras. We shall also consider nilpotent-by-abelian Lie algebras. The highlights are listed in the next paragraph. An explanation of unfamiliar terms can be found below in Section 2.

Main Results *The following statements hold for any Lie algebra L and its derived subalgebra L'.*

1. *Whether or not L is nilpotent is determined by $U(L)$. In the case when L is nilpotent, both its minimal number of generators and its nilpotence class are determined. If L is nilpotent of class at most two then L itself is determined by $U(L)$.*
2. *The quotient L'/L'' is determined by $U(L)$. If the dimension of L'/L'' is finite then the following statements also hold. Whether or not L' is nilpotent is determined by $U(L)$. In the case when L' is nilpotent, both its minimal number of generators and its nilpotence class are determined. In particular, whether or not L is metabelian is determined.*

The proof of these results can be found in Sections 2 through 7. Inspired by these results, one might hope that all finite-dimensional nilpotent or metabelian Lie algebras are determined by their universal envelope. However, this is not the case - at least in positive characteristic - as seen by the following example (see also a similar example by Kuznetsov in [7]):

Example B. *Let $A = \mathbb{F}x_0 + \cdots + \mathbb{F}x_p$ be an abelian Lie algebra over a field $\mathbb{F}$ of characteristic p. Consider the Lie algebras $L = A + \mathbb{F}\lambda + \mathbb{F}\pi$ and $H = A + \mathbb{F}\lambda + \mathbb{F}z$ with relations given by $[\lambda, x_i] = x_{i-1}$, $[\pi, x_i] = x_{i-p}$, $[\lambda, \pi] = [z, H] = 0$, and $x_i = 0$ for every $i < 0$. Note that L and H are each metabelian and nilpotent of class $p+1$. Further notice that the centre of L is spanned by x_0 while the centre of H is spanned by z and x_0; so, L and H are not isomorphic. However, using the PBW theorem, it is easy to see that the Lie homomorphism $\Phi : L \to U(H)$ defined by $\Phi_{|A+\mathbb{F}\lambda} = id$, $\Phi(\pi) = z + \lambda^p$ can be extended to an algebra isomorphism $U(L) \to U(H)$.*

In fact, we will see in Section 8 that the natural Hopf algebra structures of $U(L)$ and $U(H)$ are isomorphic. Along this same line, we shall also take a closer look at Example A in order to prove that the minimal number of generators of the Lie algebra L is not determined by the Hopf algebra structure of $U(L)$. In sharp contrast, the enriched Hopf algebra structure of $U(L)$ is known to completely determine any Lie algebra L over a field of characteristic zero. We stress that, in spite of all this, the characteristic zero case of the isomorphism problem remains entirely open.

In Section 9, we show that the universal enveloping algebra of a Lie superalgebra L does not determine L even when L is 1-dimensional. In the final section, we use one of our intermediate results to derive a new proof of a theorem of Bahturin (see [1]): every nilpotent-by-abelian Lie algebra can be embedded into an associative envelope satisfying a polynomial identity.

2. Preliminaries

Throughout this paper, L denotes a Lie algebra with an ordered basis $\{x_j\}_{j\in\mathcal{J}}$ over a field $\mathbb{F}$. We also fix a second Lie algebra H with the property that $U(L) \cong U(H)$.

Let $\gamma_1(L) := L$. We denote by $\gamma_n(L) := [\gamma_{n-1}(L), L]$ the n-th term of the lower central series of L. Hence, $L' = \gamma_2(L)$. Recall that L is said to be nilpotent if $\gamma_n(L) = 0$ for some n; the nilpotence class of L is the minimal integer c such that $\gamma_{c+1}(L) = 0$. Also recall that L is called metabelian if L' is abelian, whereas, L is called nilpotent-by-abelian whenever L' is nilpotent.

The Poincaré-Birkhoff-Witt (PBW) theorem (see [1] or [13], for example) allows us to view L as a Lie subalgebra of $U(L)$ in such a way that $U(L)$ has a basis consisting of monomials of the form $x_{j_1}^{a_1} \cdots x_{j_t}^{a_t}$ where $j_1 \leq \cdots \leq j_t$ are in $\mathcal{J}$ and t and each a_i are non-negative integers. The augmentation map $\varepsilon_L : U(L) \to \mathbb{F}$ (with respect to L) is induced by $\varepsilon_L(x_j) = 0$ for every $j \in \mathcal{J}$. Clearly ε_L depends only on L and not on our particular choice of basis. We shall denote the kernel of ε_L by $\omega(L)$; thus, $\omega(L) = LU(L) = U(L)L$. For each y in $U(L)$, we define $\deg(y)$ to be the maximum (total) degree of the PBW monomials appearing in the PBW expansion of y.

Let M be an ideal of L and consider the natural epimorphism $L \to L/M$. It is well-known that this map extends to an algebra epimorphism $U(L) \to$

$U(L/M)$ with kernel $MU(L) = U(L)M$. Moreover, $L \cap MU(L) = M$ and so $L + MU(L)/MU(L) \cong L/M$.

A first natural question is whether or not the enveloping algebra of L determines $\omega(L)$. The following lemma answers this question in the positive.

Lemma 2.1. *Let L and H be Lie algebras and suppose that $\varphi : U(L) \to U(H)$ is an algebra isomorphism. Then there exists an algebra isomorphism $\psi : U(L) \to U(H)$ with the property that $\psi(\omega(L)) = \omega(H)$.*

Proof. Consider the map $\eta : L \to U(H)$ defined by $\eta = \varphi - \varepsilon_H\varphi$. It is easy to check that η is a Lie homomorphism. Hence, by universal property of $U(L)$, there exists a unique algebra homomorphism $\overline{\eta} : U(L) \to U(H)$ extending η and preserving unity. Observe next that $\overline{\eta}(\omega(L)) \subseteq \omega(H)$ since $\eta(L) \subseteq \omega(H)$. It remains to show that $\overline{\eta}$, or equivalently $\varphi^{-1}\overline{\eta}$, is an isomorphism. Let $y \in U(L)$ and express it as a linear combination of PBW monomials: $y = \sum \alpha_{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n}$, for some $\alpha_{a_1,\ldots,a_n} \in \mathbb{F}$. We have

$$\begin{aligned} \varphi^{-1}\overline{\eta}(y) &= \varphi^{-1}(\sum \alpha_{a_1,\ldots,a_n} \eta(x_1)^{a_1} \cdots \eta(x_n)^{a_n}) \\ &= \sum \alpha_{a_1,\ldots,a_n} (x_1 - \varepsilon_H\varphi(x_1))^{a_1} \cdots (x_n - \varepsilon_H\varphi(x_n))^{a_n} \\ &= y + z, \end{aligned}$$

where z is such that $\deg(z) < \deg(y)$. According to the PBW Theorem, however, y and z are linearly independent and so $\varphi^{-1}\overline{\eta}(y) = 0$ if and only if $y = 0$. Thus $\varphi^{-1}\overline{\eta}$ is injective. To see why $\varphi^{-1}\overline{\eta}$ is surjective, first notice that $\mathbb{F} \subseteq \mathrm{Im}\varphi^{-1}\overline{\eta}$ since φ and $\overline{\eta}$ preserve unity. Thus, $L \subseteq \mathrm{Im}\varphi^{-1}\overline{\eta}$ since, for every $x \in L$, we have $\varphi^{-1}\overline{\eta}(x) = \varphi^{-1}(\eta(x)) = x - \varepsilon_H\varphi(x)$. Since $\mathbb{F} \cup L$ generates $U(L)$ as an algebra, it follows that $\mathrm{Im}\varphi^{-1}\overline{\eta} = U(L)$, as required. □

Henceforth, $\varphi : U(L) \to U(H)$ denotes an algebra isomorphism that preserves the corresponding augmentation ideals.

3. Powers of the augmentation ideal

Since φ preserves $\omega(L)$, it also preserves the filtration of $U(L)$ given by the powers of $\omega(L)$:

$$U(L) = \omega^0(L) \supseteq \omega^1(L) \supseteq \omega^2(L) \supseteq \ldots.$$

Corresponding to this filtration is the graded associative algebra

$$\mathrm{gr}(U(L)) = \oplus_{i\geq 0}\omega^i(L)/\omega^{i+1}(L),$$

where the multiplication in $\mathrm{gr}(U(L))$ is induced by

$$(y_i + \omega^{i+1}(L))(z_j + \omega^{j+1}(L)) = y_i z_j + \omega^{i+j+1}(L),$$

for all $y_i \in \omega^i(L)$ and $z_j \in \omega^j(L)$. Certainly $\mathrm{gr}(U(L))$ is determined by $U(L)$.

There is an analogous construction for Lie algebras. That is, one can consider the graded Lie algebra of L corresponding to its lower central series given by $\mathrm{gr}(L) = \oplus_{i\geq 1}\gamma_i(L)/\gamma_{i+1}(L)$.

For each $y \in L$, we define the height, $\nu(y)$, of y to be the largest subscript n such that $y \in \gamma_n(L)$ if n exists and to be infinite if it does not. We shall call an ordered basis $\{x_j\}_{j\in\mathcal{J}}$ of L *homogeneous* (with respect to the lower central series of L) if $\gamma_n(L) = \langle x_j | \nu(x_j) \geq n\rangle_{\mathbb{F}}$, for every $n \geq 1$.

Theorem 3.1 ([11])**.** *Let L be a Lie algebra that admits a homogeneous basis $\{x_j\}_{j\in\mathcal{J}}$. Then the following statements hold.*

(1) *For each integer $n \geq 1$, the set of all PBW monomials of the form $x_{j_1}^{a_1}x_{j_2}^{a_2}\cdots x_{j_s}^{a_s}$ with the property that $\sum_{k=1}^{s} a_k\nu(x_{j_k}) \geq n$ forms an $\mathbb{F}$-basis for $\omega^n(L)$.*
(2) *For every positive integer n, $L \cap \omega^n(L) = \gamma_n(L)$.*
(3) $\cap_{n\geq 1}\omega^n(L) = 0$ *if and only if* $\cap_{n\geq 1}\gamma_n(L) = 0$.
(4) *The homomorphism $U(\mathrm{gr}(L)) \to \mathrm{gr}(U(L))$ induced by the natural embeddings $\gamma_n(L)/\gamma_{n+1}(L) \to \gamma_n(L) + \omega^{n+1}(L)/\omega^{n+1}(L)$ is an isomorphism of graded associative algebras.*

We remark that part (4) was proved first by Knus ([6]) in the special case when L is finite-dimensional over a field characteristic zero.

It was assumed in [11] that every Lie algebra admits a homogeneous basis. A similar oversight appears in [13], see Section 1.9. However, this is not always true. The following example shows that some finiteness condition is required to guarantee the existence of a homogeneous basis.

Example C. *Let $L = \oplus_{i\geq 1}L_i$ be a graded Lie algebra such that each L_i is finite dimensional and L is generated by L_1. For example, one can take L to be a finitely generated free Lie algebra. Let M be the Cartesian product $M := \prod_{i\geq 1} L_i$, where the multiplication in M is induced from L. Then M does not admit a homogeneous basis.*

Note that $\gamma_i(M) = \prod_{j\geq i} L_j$ and $\gamma_i(M)/\gamma_{i+1}(M) \cong L_i$, for every $i \geq 1$. Suppose, to the contrary, that M possesses a homogeneous basis X. For each $i \geq 1$, put $X_i := X \cap \gamma_i(M)$ and let W_i be the subspace of M spanned by the set difference $X_i - X_{i+1}$. Since X is a homogeneous basis, each X_i is a basis for $\gamma_i(L)$. We may embed $\gamma_i(M)/\gamma_{i+1}(M)$ as a subspace of M under the natural vector space isomorphism $\gamma_i(M)/\gamma_{i+1}(M) \cong W_i$. It is not hard to show that the induced linear map

$$\phi : \oplus_{i\geq 1}\gamma_i(M)/\gamma_{i+1}(M) \to M$$

is an embedding. Moreover, X is in the image of ϕ. So, ϕ is actually a vector space isomorphism. However, this is impossible since a basis for L is countably infinite whereas a basis for M is uncountably infinite.

Nonetheless, it is easy to see that homogeneous bases exist in two important cases: if the lower central series of L stabilizes or if L is graded over the positive integers and generated by its first degree component. In fact, we shall see below that parts (2) through (4) in Theorem 3.1 hold true for arbitrary Lie algebras. An examination of the proof of Proposition 3.1 in [11] shows that part (1) can be replaced by the following general result:

Corollary 3.2. *Let L be an arbitrary Lie algebra and let $X = \{\bar{x}_i\}_{i\in\mathcal{I}}$ be a homogeneous basis of* $\mathrm{gr}(L)$. *Take a coset representative x_i for each $\bar{x}_i$. Then the set of all PBW monomials $x_{i_1}^{a_1}x_{i_2}^{a_2}\cdots x_{i_s}^{a_s}$ with the property that $\sum_{k=1}^{s} a_k\nu(x_{i_k}) = n$ forms an $\mathbb{F}$-basis for $\omega^n(L)$ modulo $\omega^{n+1}(L)$, for every $n \geq 1$.*

Corollary 3.3. *Let L be any Lie algebra. We have $L \cap \omega^n(L) = \gamma_n(L)$, for every positive integer n.*

Proof. Since every nilpotent Lie algebra possesses a homogeneous basis, the statement holds for nilpotent Lie algebras, by part (2) of Theorem 3.1. So,

$$L/\gamma_n(L)\ \cap \omega^n(L/\gamma_n(L)) = \gamma_n(L/\gamma_n(L)) = 0.$$

Under the identification $U(L/\gamma_n(L)) = U(L)/\gamma_n(L)U(L)$, we have $L/\gamma_n(L) = L + \gamma_n(L)U(L)/\gamma_n(L)U(L)$ and $\omega^n(L/\gamma_n(L)) = \omega^n(L)/\gamma_n(L)U(L)$. Hence,

$$[L + \gamma_n(L)U(L)/\gamma_n(L)U(L)] \cap [\omega^n(L)/\gamma_n(L)U(L)] = 0.$$

This means that

$$L \cap \omega^n(L) \subseteq L \cap \gamma_n(L)U(L) = \gamma_n(L).$$

The reverse inclusion is obvious. □

A Lie algebra L is called residually nilpotent if $\cap_{n\geq 1}\gamma_n(L) = 0$; analogously, an associative ideal I of $U(L)$ is residually nilpotent whenever $\cap_{n\geq 1} I^n = 0$.

Lemma 3.4. *Let L be a residually nilpotent Lie algebra. For every finite linearly independent subset $\{x_1, \ldots, x_t\}$ of L there exists a positive integer N such that $x_1, \ldots, x_t$ are linearly independent modulo $\gamma_N(L)$.*

Proof. Suppose, to the contrary, that $x_1, \ldots, x_t$ are linearly dependent modulo $\gamma_n(L)$, for every $n \geq 1$. Without loss of generality, we can assume that t is minimal in the sense that there exists an integer N such that each of the finitely many proper subsets of $\{x_1, \ldots, x_t\}$ is linearly independent modulo $\gamma_N(L)$. By assumption, for every $n \geq N$, there exist coefficients $\alpha_{1,n}, \ldots, \alpha_{t,n} \in \mathbb{F}$, not all zero, such that

$$v_n := \alpha_{1,n}x_1 + \alpha_{2,n}x_2 + \cdots + \alpha_{t,n}x_t \in \gamma_n(L).$$

Notice that, by our choice of N, none of the coefficients $\alpha_{i,n}$ are zero. So, without loss of generality, we assume that $\alpha_{1,N} = 1$. Now we have $\alpha_{1,n}v_N - v_n \in \gamma_N(L)$, for every $n > N$. But $\alpha_{1,n}v_N - v_n = \beta_2x_2 + \cdots + \beta_tx_t$, for some $\beta_2, \ldots, \beta_t \in \mathbb{F}$. It follows that $\alpha_{1,n}v_N - v_n = 0$, for every $n > N$. Hence, $v_N = \alpha_{1,n}^{-1}v_n \in \gamma_n(L)$, for every $n \geq 1$, contradicting our assumption that L is residually nilpotent. □

Corollary 3.5. *Let L be a Lie algebra. Then L is residually nilpotent as a Lie algebra if and only if $\omega(L)$ is residually nilpotent as an associative ideal.*

Proof. Clearly $\cap_{n\geq 1}\gamma_n(L) \subseteq \cap_{n\geq 1}\omega^n(L)$; hence, sufficiency holds. In order to prove necessity, suppose that L is residually nilpotent but there exists non-zero z in $\cap_{n\geq 1}\omega^n(L)$. Then there is a finite number of basis elements $x_{j_1}, \ldots, x_{j_t}$ of L such that $z = \sum \alpha x_{j_1}^{a_1} \cdots x_{j_t}^{a_t}$. By Lemma 3.4, there exists a positive integer N such that $x_{j_1}, \ldots, x_{j_t}$ are linearly independent modulo $\gamma_N(L)$. Now consider the natural homomorphism $\phi : U(L) \to U(L/\gamma_N(L))$. Since $L/\gamma_N(L)$ admits a homogeneous basis, we have $\phi(\cap_{n\geq 1}\omega^n(L)) \subseteq \cap_{n\geq 1}\omega^n(L/\gamma_N(L)) = 0$, by Theorem 3.1, part (3). Therefore, $\phi(z) = \phi(\sum \alpha x_{j_1}^{a_1} \cdots x_{j_t}^{a_t}) = \sum \alpha(x_{j_1} + \gamma_N(L))^{a_1} \cdots (x_{j_t} + \gamma_N(L))^{a_t} = 0$. However, $x_{j_1} + \gamma_N(L), \ldots, x_{j_t} + \gamma_N(L)$ are linearly independent, and so each $\alpha = 0$ by the PBW Theorem. Hence, $z = 0$, a contradiction. □

Finally, we generalise part (4) of Theorem 3.1. The map

$$\gamma_n(L)/\gamma_{n+1}(L) \to \gamma_n(L) + \omega^{n+1}(L)/\omega^{n+1}(L)$$

is a well-defined vector space embedding by Corollary 3.3. It is easy to check that this induces a Lie algebra embedding $\mathrm{gr}(L) \to \mathrm{gr}(U(L))$ which extends uniquely to an associative algebra homomorphism $\phi : U(\mathrm{gr}(L)) \to \mathrm{gr}(U(L))$. Because $\phi(\gamma_1(L)/\gamma_2(L)) = L + \omega^2(L)/\omega^2(L) = \omega^1(L)/\omega^2(L)$ generates $\mathrm{gr}(U(L))$ as an associative algebra, it follows that ϕ is surjective. The fact that ϕ is injective is easily deduced from Corollary 3.2.

Corollary 3.6. *For any Lie algebra L, the map $\phi : U(\mathrm{gr}(L)) \to \mathrm{gr}(U(L))$ is an isomorphism of graded associative algebras.*

4. Nilpotent Lie algebras

Proposition 4.1. *The graded Lie algebra $\mathrm{gr}(L)$ is determined by $U(L)$.*

Proof. By Corollary 3.6, we can embed $\mathrm{gr}(L)$ into $\mathrm{gr}(U(L))$. Under this identification, $\gamma_1(L)/\gamma_2(L) = \omega^1(L)/\omega^2(L)$. Consequently,

$$\mathrm{gr}(L) = \langle \gamma_1(L)/\gamma_2(L)\rangle_{\mathrm{Lie}} = \langle \omega^1(L)/\omega^2(L)\rangle_{\mathrm{Lie}}$$

is determined by $U(L)$. □

Corollary 4.2. *For each pair of integers (m, n) such that $n \geq m \geq 1$, the quotient $\gamma_n(L)/\gamma_{m+n}(L)$ is determined by $U(L)$.*

Proof. An easy calculation shows that $\gamma_n(\mathrm{gr}(L)) = \oplus_{i\geq n}\gamma_i(L)/\gamma_{i+1}(L)$. Thus, each quotient $\gamma_n(L)/\gamma_{n+1}(L) \cong \gamma_n(\mathrm{gr}(L))/\gamma_{n+1}(\mathrm{gr}(L))$ is determined by $U(L)$. Since abelian Lie algebras of the same (possibly infinite) dimension are isomorphic, it follows that each $\gamma_n(L)/\gamma_{n+m}(L)$ is also determined. □

We are now ready for our first main result.

Proposition 4.3. *The following statements hold for every Lie algebra L.*

1. *Whether or not L is residually nilpotent is determined by $U(L)$.*
2. *Whether or not L is nilpotent is determined by $U(L)$.*
3. *If L is nilpotent then the nilpotence class of L is determined by $U(L)$.*
4. *If L is nilpotent then the minimal number of generators of L is determined by $U(L)$.*
5. *If L is a finitely generated free nilpotent Lie algebra then L is determined by $U(L)$.*

Proof. Part (1) is the conclusion of Corollary 3.5. Suppose now that L is nilpotent of class c. Then $\gamma_{c+1}(\mathrm{gr}(L)) = \oplus_{i\geq c+1}\gamma_i(L)/\gamma_{i+1}(L) = 0$. Hence, by Proposition 4.1, $\gamma_{c+1}(\mathrm{gr}(H)) = 0$; in other words, $\gamma_{c+1}(H) = \gamma_{c+2}(H)$. But L, and hence H, is residually nilpotent by part (1). Thus $\gamma_{c+1}(H) = 0$, proving (2) and (3). It is well-known that the minimal number of generators of a nilpotent Lie algebra L is exactly $\dim_{\mathbb{F}}(L/L')$. But L/L' is determined by $U(L)$, as was shown in Corollary 4.2. This proves (4). Finally suppose that L is finitely generated free nilpotent and $U(L) \cong U(H)$. Then, by parts (3) and (4), there exists a Lie epimorphism from L to H. Hence, L and H are isomorphic since they have the same finite dimension. □

5. Nilpotent Lie algebras of class at most two

In this section, we adapt a group ring technique from [10] in order to show that all nilpotent Lie algebras of class at most 2 are determined by their enveloping algebras.

First, for each $n \geq 1$, let us fix a subspace $K_n(L)$ of $\omega^n(L)$ such that

$$\omega^n(L)/\omega^{n+1}(L) = \gamma_n(L) + \omega^{n+1}(L)/\omega^{n+1}(L) \oplus K_n(L)/\omega^{n+1}(L)$$

is vector space decomposition.

Lemma 5.1. *For every $n \geq 1$, the following statements hold.*

1. *$K_n(L)$ is an ideal of $U(L)$.*
2. *$\gamma_n(L) + \omega^{n+1}(L) = \gamma_n(\omega(L)) + \omega^{n+1}(L)$.*
3. *$\varphi(\gamma_n(L) + \omega^{n+1}(L)) = \gamma_n(H) + \omega^{n+1}(H)$.*
4. *$\omega^n(H)/\omega^{n+1}(H) = \gamma_n(H)+\omega^{n+1}(H)/\omega^{n+1}(H)\oplus\varphi(K_n(L))/\omega^{n+1}(H)$.*
5. *$\varphi(\gamma_n(L) + K_{n+1}(L)) = \gamma_n(H) + \varphi(K_{n+1}(L))$.*

Proof. Since $\omega^{n+1}(L) \subseteq K_n(L) \subseteq \omega^n(L)$ by definition, certainly (1) holds. Part (2) follows from repeated application of the identity $[xy, z] = x[y, z] + [x, z]y$. Whence (3) and subsequently (4) follow. In order to prove (5), let $x \in \gamma_n(L)$. Then $\varphi(x) = y + z \in \gamma_n(H) + \omega^{n+1}(H)$ by part (3). But, by part (4), we have $z = u+v \in \gamma_{n+1}(H)+\varphi(K_{n+1}(L))$. Thus, $\varphi(x) = (u+y)+v \in \gamma_n(H) + \varphi(K_{n+1}(L))$, and we have proved $\varphi(\gamma_n(L) + K_{n+1}(L)) \subseteq \gamma_n(H) + \varphi(K_{n+1}(L))$. In order to prove the reverse inclusion, one can employ a similar argument to show that $\varphi^{-1}(\gamma_n(H) + \varphi(K_{n+1}(L))) \subseteq \gamma_n(L) + K_{n+1}(L)$, as required. □

Proposition 5.2. *For every $n \geq 1$, we have the following isomorphisms of Lie algebras.*

(1) $\gamma_n(L)/\gamma_{n+2}(L) \cong \gamma_n(L) + K_{n+1}(L)/K_{n+1}(L)$.
(2) $\gamma_n(H)/\gamma_{n+2}(H) \cong \gamma_n(H) + \varphi(K_{n+1}(L))/\varphi(K_{n+1}(L))$.
(3) $\gamma_n(L)/\gamma_{n+2}(L) \cong \gamma_n(H)/\gamma_{n+2}(H)$.

In particular, $L/\gamma_3(L) \cong H/\gamma_3(H)$.

Proof. It follows from part (1) of the previous lemma that the natural map $\Psi : \gamma_n(L) \to \gamma_n(L) + K_{n+1}(L)/K_{n+1}(L)$ is a well-defined Lie epimorphism. We now compute its kernel. Indeed, Corollary 3.3,

$$\ker(\Psi) = \gamma_n(L) \cap K_{n+1}(L) \subseteq \gamma_n(L) \cap \omega^{n+1}(L) = \gamma_{n+1}(L);$$

therefore, $\ker(\Psi) = \gamma_{n+1}(L) \cap K_{n+1}(L) = \gamma_{n+1}(L) \cap \omega^{n+2}(L) = \gamma_{n+2}(L)$. This proves part (1). The proof of (2) is similar. Next, notice that, by part (5) of the previous lemma, φ induces an isomorphism

$$\gamma_n(L) + K_{n+1}(L)/K_{n+1}(L) \to \gamma_n(H) + \varphi(K_{n+1}(L))/\varphi(K_{n+1}(L)).$$

Thus, (3) follows after reviewing parts (1) and (2). □

We can now deduce the main result of this section.

Corollary 5.3. *Every nilpotent Lie algebra of class at most two is determined by its enveloping algebra.*

Proof. Suppose that L is nilpotent of class $c \leq 2$. Then H is also nilpotent of class c by Proposition 4.3. Proposition 5.2 now yields that $L \cong L/\gamma_3(L) \cong H/\gamma_3(H) \cong H$, as required. □

6. Subalgebra correspondence

Throughout this section, S denotes a subalgebra of L. We fix a basis $\{x_i\}_{i\in\mathcal{I}}$ of S and extend this basis to an ordered basis $\{x_i\}_{i\in\mathcal{I}} \cup \{y_k\}_{k\in\mathcal{K}}$ of L, where the x_i's are less than the y_k's. So, a typical PBW monomial in $U(L)$ has the form $x_{i_1}^{a_1} \cdots x_{i_m}^{a_m} y_{k_1}^{b_1} \cdots y_{k_n}^{b_n}$.

Proposition 6.1. *Let S be any subalgebra of L. The following statements hold for every integer $n \geq 1$.*

(1) $\omega(S) \cap \omega^n(S)\omega(L) = \omega^{n+1}(S)$; *hence,* $L \cap \omega^n(S)\omega(L) = \gamma_{n+1}(S)$.
(2) $\omega(S) \cap \omega^n(S)U(L) = \omega^n(S)$; *hence,* $L \cap \omega^n(S)U(L) = \gamma_n(S)$.
(3) $\omega^n(S)/\omega^{n+1}(S)$ *embeds into* $\omega^n(S)U(L)/\omega^n(S)\omega(L)$.
(4) $\omega^n(S)/\omega^{n+1}(S)$ *embeds into* $\omega^n(S)U(L)/\omega^{n+1}(S)U(L)$.

Proof. Obviously, $\omega^{n+1}(S) \subseteq \omega(S) \cap \omega^n(S)\omega(L)$. Let z be a non-zero element in $\omega(S) \cap \omega^n(S)\omega(L)$. Then $z = \sum st$, where each $s \in \omega^n(S)$ and each t is a non-trivial PBW monomial of the form $x_{i_1}^{a_1} \cdots x_{i_m}^{a_m} y_{k_1}^{b_1} \cdots y_{k_n}^{b_n}$. So each st has the form $st = sx_{i_1}^{a_1} \cdots x_{i_m}^{a_m} y_{k_1}^{b_1} \cdots y_{k_n}^{b_n}$, where either $b_1 + \cdots + b_n \neq 0$ or $b_1 + \cdots + b_n = 0$ and $a_1 + \cdots + a_m \neq 0$. Now we have

$$z = \sum \alpha uv + \sum \beta w,$$

where each u is a PBW monomial in $\omega(S)$, each v is a non-trivial PBW monomial in the y_k's only, and each w is a PBW monomial in $\omega(S)$ such that $\sum \beta w \in \omega^{n+1}(S)$. This is the unique PBW representation of z. However, $z \in \omega(S)$ and therefore, by the linear independence of PBW monomials, $z = \sum \beta w \in \omega^{n+1}(S)$. This proves the first assertion in part (1). To prove the second assertion, let $z \in L \cap \omega^n(S)\omega(L) \subseteq L \cap \omega(S)\omega(L)$. So, $z = \sum st$, where each $s \in \omega(S)$ and each t is a (possibly trivial) monomial in the y_k's only. In fact all the t's must be trivial since elements of L are of degree one. It follows that $z \in \omega(S)$ and the result now follows from the first assertion and Corollary 3.4. To prove (2), let $z \in \omega(S) \cap \omega^n(S)U(L)$. Then $z = r + s \in \omega^n(S) + \omega^n(S)\omega(L)$. Thus, $s = z - r \in \omega(S) \cap \omega^n(S)\omega(L) = \omega^{n+1}(S)$, by part (1). Hence $z \in \omega^n(S)$, yielding the first assertion in (2). The second assertion follows as above. Parts (3) and (4) are simple consequences of (1) and (2). □

Corollary 6.2. *We have $S/S' \cong \omega(S)U(L)/\omega(S)\omega(L)$. Consequently, if T is a subalgebra of H such that $\varphi(SU(L)) = TU(H)$ then $S/S' \cong T/T'$.*

Proof. By previous proposition, we know that $S/S' \cong \omega(S)/\omega^2(S)$ embeds into $\omega(S)U(L)/\omega(S)\omega(L)$. But, $\omega(S)U(L) = \omega(S) + \omega(S)\omega(L)$, so this embedding is an isomorphism. Now notice that $\omega(S)U(L) = SU(L)$ and $\omega(S)\omega(L) = (SU(L))\omega(L)$. Therefore,

$$\varphi(\omega(S)U(L)) = \varphi(SU(L)) = TU(H) = \omega(T)U(H)$$

and

$$\varphi(\omega(S)\omega(L)) = \varphi(SU(L))\varphi(\omega(L)) = \omega(T)\omega(H).$$

It follows that $S/S' \cong T/T'$, as required. □

Proposition 6.3. *The following conditions are equivalent for every subalgebra S of L.*

(1) $\cap_{n\geq 1}\gamma_n(S) = 0$.
(2) $\cap_{n\geq 1}\omega^n(S) = 0$.
(3) $\cap_{n\geq 1}\omega^n(S)U(L) = 0$.

Proof. Corollary 3.5 informs us that conditions (1) and (2) are equivalent, and certainly (3) implies (1). It remains then to prove that (2) implies (3). The PBW representation of any element z in $U(L)$ has the form $z = \sum \alpha uv$, where each u is a PBW monomial in terms of the x_i's only and each v is a PBW monomial in terms of the y_k's only. Now suppose that z is a non-zero element in $\cap_{n\geq 1}\omega^n(S)U(L)$. We may factor the PBW representation of z into the form $z = \sum(\sum \beta u)v$, where all the v's are distinct. We will prove that for each v the corresponding element $\sum \beta u$ lies in $\cap_{n\geq 1}\omega^n(S)$. So, fix n and regard $z \in \omega^n(S)U(L)$. Then $z = \sum st$, where each $s \in \omega^n(S)$ and each t is a (possibly trivial) PBW monomial in the y_k's only. Without loss of generality, we assume that the t's are distinct. It now follows from the uniqueness of the PBW representation of z that each t is equal to some v

and $\sum \beta u = s \in \omega^n(S)$. Since n was arbitrary, it follows that each $\sum \beta u$ lies in $\cap_{n\geq 1}\omega^n(S)$. Hence (2) implies (3) and the proof is complete. □

Now let I be an ideal of L. Recall that the kernel of the natural map $U(L) \to U(L/I)$ is equal to $IU(L) = U(L)I$. It follows that $\omega^n(I)U(L) = I^nU(L) = (IU(L))^n$, for every $n \geq 1$, where I^n is the vector subspace of $\omega(L)$ spanned by all elements of the form $z_1z_2\cdots z_n$, where $z_i \in I$.

Corollary 6.4. *The Lie ideal I is residually nilpotent if and only if the associative ideal $IU(L)$ is residually nilpotent.*

It is easy to see that every enveloping algebra $U(L)$ has the invariant dimension property; in other words, the rank of every free $U(L)$-module is uniquely defined. We shall use the fact that $U(L/I)$ has the invariant dimension property below. Next observe that each quotient $I^nU(L)/I^{n+1}U(L)$ has a natural $U(L/I) \cong U(L)/IU(L)$-module structure given by

$$(u + I^{n+1}U(L)) \cdot (z + IU(L)) = uz + I^{n+1}U(L),$$

for every $u \in I^nU(L)$ and $z \in U(L)$.

For the remainder of this section, we fix a homogeneous basis $\{\bar{x}_i\}_{i\in\mathcal{I}}$ of $\mathrm{gr}(I)$ and take a fixed coset representative x_i for each $\bar{x}_i$. So, $\{x_i\}_{i\in\mathcal{I}}$ is a linearly independent subset of I that can be extended to an ordered basis X of I. Finally, we extend X to an ordered basis $X \cup \{y_k\}_{k\in\mathcal{K}}$ of L, where the elements in X are less than the y_k's.

Lemma 6.5. *Let I be an ideal of L. Then each factor $I^nU(L)/I^{n+1}U(L)$ is a free $U(L/I)$-module with rank of equal to the vector space dimension of $\omega^n(I)/\omega^{n+1}(I)$.*

Proof. By Corollary 3.2, we know that $\omega^n(I)/\omega^{n+1}(I)$ has a basis of the form $\{u_m + \omega^{n+1}(I)\}_{m\in\mathcal{M}}$, where the u_m's are PBW monomials (involving only the x_i's with $i \in \mathcal{I}$). We claim that $\{u_m + I^{n+1}U(L)\}_{m\in\mathcal{M}}$ is a $U(L/I)$-basis for $I^nU(L)/I^{n+1}U(L)$. The fact that it is a $U(L/I)$-generating set is clear. It remains to show $U(L/I)$-independence. If we suppose to the contrary, then there exist elements $m_1, \ldots, m_t \in \mathcal{M}$, scalars $\alpha_1, \ldots, \alpha_t$, and PBW monomials $v_1, \ldots, v_t \in U(L)$ with non-trivial images in $U(L/I)$ such that

$$z := \alpha_1 u_{m_1} v_1 + \cdots + \alpha_t u_{m_t} v_t \in I^{n+1}U(L).$$

Notice that the restriction on the v_j's is equivalent to each v_j being a PBW monomial in the y_k's only. Collecting terms, we may write z in the form $z = \sum(\sum \beta u_{m_i})v_j$, where the v_j's are distinct. However, $z \in I^{n+1}U(L) = \omega^{n+1}(I)U(L)$. So, $z = \sum st$, where each $s \in \omega^{n+1}(I)$, and each t is a (possibly trivial) PBW monomial in the y_k's only. Collecting terms again allows us to assume that the t's are distinct. It now follows from the uniqueness of the PBW representation of z that each t is equal to some v_j and the corresponding element $\sum \beta u_{m_i}$ of each v_j is equal to some $s \in \omega^{n+1}(I)$. This contradicts with the fact that u_m's are a basis for $\omega^n(I)$ modulo $\omega^{n+1}(I)$.

It remains to show that the rank of $I^nU(L)/I^{n+1}U(L)$ coincides with the dimension of $\omega^n(I)/\omega^{n+1}(I)$. Certainly

$$\begin{aligned}\mathrm{rank}_{U(L/I)} I^nU(L)/I^{n+1}U(L) &= |\{u_m + I^{n+1}U(L)\}_{m\in\mathcal{M}}| \\ &\leq |\{u_m + \omega^{n+1}(I)\}_{m\in\mathcal{M}}| \\ &= \dim_{\mathbb{F}}\omega^n(I)/\omega^{n+1}(I).\end{aligned}$$

Thus, if equality did not hold then $u_{m_1} + I^{n+1}U(L) = u_{m_2} + I^{n+1}U(L)$ for some distinct $m_1, m_2 \in \mathcal{M}$. However, then $z := u_{m_1} - u_{m_2} \in I^{n+1}U(L)$ and so (arguing as above) $u_{m_1} - u_{m_2} \in \omega^{n+1}(I)$, a contradiction. □

For each integer $n \geq 1$, put $c_n = c_n(I) = \dim_{\mathbb{F}}\omega^n(I)/\omega^{n+1}(I)$ and $d_n = d_n(I) = \dim_{\mathbb{F}}\gamma_n(I)/\gamma_{n+1}(I)$. Observe that if $c_1 = d_1$ is finite then every c_n and d_n is finite. Indeed, let $A := \oplus_{n\geq 1}\omega^n(I)/\omega^{n+1}(I)$. Then A is c_1-generated and so the dimension $c_1 + \cdots + c_n$ of each nilpotent quotient A/A^{n+1} is finite.

Proposition 6.6. *Suppose $d_1(I)$ is finite and suppose J is an ideal of H such that $\varphi(IU(L)) = JU(H)$. Then the following statements hold.*

(1) $c_n(I) = c_n(J)$, *for every* $n \geq 1$.
(2) $d_n(I) = d_n(J)$, *for every* $n \geq 1$.
(3) $\gamma_n(I)/\gamma_{m+n}(I) \cong \gamma_n(J)/\gamma_{m+n}(J)$, *for each pair of integers* (m, n) *such that* $n \geq m \geq 1$.
(4) *If I is nilpotent then J is nilpotent of the same class.*

Proof. Since $\varphi(IU(L)) = JU(H)$, we have $\varphi(I^nU(L)) = \varphi((IU(L))^n) = (JU(H))^n = J^nU(H)$, and so $I^nU(L)/I^{n+1}U(L) \cong J^nU(L)/J^{n+1}U(L)$. Now (1) follows from Lemma 6.5. In particular, $d_1 = c_1 = c_1(J) = d_1(J)$. In order to prove (2), first notice that it follows from Corollary 3.2 that

$$c_{n+1} = d_{n+1} + \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n+1} \prod_{i=1}^{n} \binom{d_i + \lambda_i - 1}{\lambda_i},$$

for each $n \geq 1$, where the λ_i's are non-negative integers. Therefore, the sequence $c_1, c_2, \ldots$ is uniquely determined by the sequence $d_1, d_2, \ldots$, and conversely. Thus (2) follows from (1). Part (3) follows as in the proof of Corollary 4.2. To prove (4), suppose that I is nilpotent of class c. Then, $\gamma_{c+1}(J) = \gamma_{c+2}(J)$, by part (2). However J is residually nilpotent by Corollary 6.4. Consequently, $\gamma_{c+1}(J) = 0$. □

7. Nilpotent-by-abelian Lie algebras

We remark that the commutator ideal $L'U(L) = [\omega(L), \omega(L)]U(L)$ of $U(L)$ is preserved by φ. Therefore, the results of the previous section can be applied to the case $I = L'$ and $J = H'$.

Corollary 7.1. *Let L be any Lie algebra. Then the following statements hold.*

(1) *The quotient L'/L'' is determined by $U(L)$.*
(2) *Whether or not L' is residually nilpotent is determined by $U(L)$.*

Corollary 7.2. *Let L be any Lie algebra such that L'/L'' is finite dimensional. Then the following statements hold.*

(1) *For all integers $n \geq m \geq 1$, the quotients $\gamma_n(L')/\gamma_{m+n}(L')$ are determined by $U(L)$.*
(2) *Whether or not L' is nilpotent is determined by $U(L)$. In the case L' is nilpotent, the nilpotence class and the minimal number of generators of L' are each determined by $U(L)$. In particular, whether or not L is metabelian is determined by $U(L)$.*

Put $\delta_1(L) = L$ and denote by $\delta_{n+1}(L) = [\delta_n(L), \delta_n(L)]$ the n-th term of the derived series of L. Recall that L is said to be soluble if $\delta_n(L) = 0$ for some n; the derived length of L is the minimal integer l such that $\delta_l(L) = 0$.

Corollary 7.3. *Let L be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero. Then whether or not L is soluble is determined by $U(L)$.*

Proof. Suppose that L is soluble and let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. Set $\bar{L} := L \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Then $\dim_{\bar{\mathbb{F}}}(\bar{L}) = \dim_{\mathbb{F}}(L)$ and $\bar{L}$ has the same derived length as L. Now consider the adjoint representation of $\bar{L}$, ad: $\bar{L} \to \mathrm{gl}(\bar{L})$. Since the kernel of ad is the centre, $Z(\bar{L})$, of $\bar{L}$, we have $\bar{L}/Z(\bar{L}) \cong \mathrm{ad}(\bar{L}) \subseteq \mathrm{gl}(\bar{L})$. Thus, according to Lie's theorem, we can embed $\bar{L}/Z(\bar{L})$ into upper triangular matrices. This proves that $\bar{L}/Z(\bar{L})$ is nilpotent-by-abelian; consequently, so is L. Applying the previous corollary now shows that H is nilpotent-by-abelian; in particular, H is soluble. □

Actually, Corollary 7.3 holds in arbitrary characteristic for, according to [14], the enveloping algebra of a finite-dimensional Lie algebra can be embedded into a (Jacobson) radical algebra if and only if L is soluble.

Proposition 7.4. *If L is a finite-dimensional Lie algebra over a field of any characteristic, then whether or not L is soluble is determined by $U(L)$.*

It is an interesting problem to decide whether or not the derived length of L is also determined.

8. Enveloping algebras as Hopf algebras

Because enveloping algebras are Hopf algebras, it also makes sense to consider an enriched form of the isomorphism problem that takes this Hopf structure into account.

Recall that a bialgebra is a vector space $\mathcal{H}$ over a field $\mathbb{F}$ endowed with an algebra structure $(\mathcal{H}, M_{\mathcal{H}}, u_{\mathcal{H}})$ and a coalgebra structure $(\mathcal{H}, \Delta_{\mathcal{H}}, \epsilon_{\mathcal{H}})$ such that $\Delta_{\mathcal{H}}$ and $\epsilon_{\mathcal{H}}$ are algebra homomorphisms. A bialgebra $\mathcal{H}$ having an antipode $S_{\mathcal{H}}$ is called a Hopf algebra. It is well-known that the universal enveloping algebra of a Lie algebra is a Hopf algebra, see for example [2] or [9]. Indeed, the counit $\epsilon_{U(L)}$ is the augmentation map introduced in Section

2 and the coproduct $\Delta_{U(L)}$ is induced by $x \mapsto x \otimes 1 + 1 \otimes x$, for every $x \in L$. An explicit description of $\Delta_{U(L)}$ can be given in terms of a PBW basis of $U(L)$ (see, for example, Lemma 5.1 in Section 2 of [13]). The antipode $S_{U(L)}$ is induced by $x \mapsto -x$, for every $x \in L$. The following proposition is well-known (see Theorems 2.10 and 2.11 in Chapter 3 of [2], for example).

Proposition 8.1. *Let L and H be Lie algebras over a field $\mathbb{F}$ of characteristic $p \geq 0$.*

(1) *If $p = 0$ then the set of primitive elements of $U(L)$ is L. Thus, any Hopf algebra isomorphism $U(L) \to U(H)$ restricts to a Lie algebra isomorphism $L \to H$. Conversely, any Lie algebra isomorphism $L \to H$ induces a Hopf algebra isomorphism $U(L) \to U(H)$.*
(2) *If $p > 0$ then the set of primitive elements of $U(L)$ is L_p, the restricted Lie subalgebra of $U(L)$ generated by L. Thus, any Hopf algebra isomorphism from $U(L) \to U(H)$ restricts to a restricted Lie algebra isomorphism $L_p \to H_p$. Conversely, any restricted Lie algebra isomorphism $L_p \to H_p$ induces a Hopf algebra isomorphism $U(L) \to U(H)$.*

Observe that part (1) completely settles the characteristic zero case – assuming that the Hopf algebra structure of $U(L)$ is taken into account.

Turning to the positive characteristic case, we shall require two more basic facts.

Lemma 8.2. *Let L be a Lie algebra over any field $\mathbb{F}$ and let M be an ideal of L. Then $MU(L)$ is a Hopf ideal of $U(L)$.*

Proof. We need to show that $MU(L)$ is a coideal; in other words,

$$\Delta_{U(L)}(MU(L)) \subseteq U(L) \otimes_{\mathbb{F}} MU(L) + MU(L) \otimes_{\mathbb{F}} U(L).$$

Since $\Delta_{U(L)}$ is $\mathbb{F}$-linear, it suffices to show this on an $\mathbb{F}$-basis of $MU(L)$. So, extend a basis $\{x_i\}_{i\in\mathcal{I}}$ of M to an ordered basis $\{x_i\}_{i\in\mathcal{I}} \cup \{y_k\}_{k\in\mathcal{K}}$ of L, with the property that each x_i is less than every y_k. It follows from the PBW Theorem that $MU(L)$ has a basis consisting of the PBW monomials of the form

$$x_{i_1}^{a_1} \cdots x_{i_m}^{a_m} y_{k_1}^{b_1} \cdots y_{k_n}^{b_n},$$

where each submonomial $x_{i_1}^{a_1} \cdots x_{i_m}^{a_m}$ is nontrivial. The assertion now follows easily from the aforementioned description of $\Delta_{U(L)}$. □

Lemma 8.3. *Let L be a Lie algebra over a field $\mathbb{F}$ and let M be an ideal of L. Then, $U(L)/MU(L) \cong U(L/M)$, as Hopf algebras.*

Proof. Consider the natural epimorphism $\theta : L \to L/M$ and its extension $\bar{\theta} : U(L) \to U(L/M)$. It is clear that $S_{U(L/M)}\bar{\theta} = \bar{\theta}S_{U(L)}$. Since the kernel $MU(L)$ of $\bar{\theta}$ is a Hopf ideal of $U(L)$, as seen by Lemma 8.2, we only need to check that $\bar{\theta}$ is a coalgebra homomorphism, that is,

$$\Delta_{U(L/M)}\bar{\theta} = (\bar{\theta} \otimes \bar{\theta})\Delta_{U(L)}.$$

But clearly $\Delta_{U(L/M)}\bar{\theta}(x) = (\bar{\theta}\otimes\bar{\theta})\Delta_{U(L)}(x)$, for every $x \in L$. Since $\Delta_{U(L/M)}, \bar{\theta}$ and $\Delta_{U(L)}$ are algebra homomorphisms and L generates $U(L)$ as an algebra, the proof is complete. □

We are now ready to adapt Example A to the setting of Hopf algebras. Recall that the universal enveloping algebra of the free Lie algebra $L(X)$ on a set X is the free associative algebra $A(X)$ on X.

Example A′. *Let $\mathbb{F}$ be a field of odd characteristic p and let $L(X)$ be the free Lie algebra on $X = \{x, y, z\}$ over $\mathbb{F}$. Set $h = x + [y, z] + (ad\ x)^p(z) \in L(X)$ and put $L = L(X)/\langle h\rangle$, where $\langle h\rangle$ denotes the ideal generated by h in $L(X)$. Then L is not a free Lie algebra. There exists, however, a Hopf algebra isomorphism between $U(L)$ and the 2-generator free associative algebra. Furthermore, the minimal number of generators required to generate L is 3.*

Indeed, by Lemma 8.3, we have

$$U(L) \cong U(L(X))/\langle h\rangle U(L(X)) = A(X)/\langle h\rangle A(X),$$

as Hopf algebras. Setting $u = y+x^p$ and $v = z$ in $A(X)$, we find that $\{u, v, h\}$ freely generates $A(X)$. Now let $H = L(u, v)$, so that $U(H) = A(u, v)$. The map $A(X) \to U(H)$ given by $u \mapsto u$, $v \mapsto v$ and $h \mapsto 0$ is a bialgebra homomorphism. Consequently, $U(L) \cong U(H)$ as Hopf algebras, even though $L \ncong H$ as Lie algebras. Now suppose, to the contrary, that there exist $a, b \in L$ such that $L = \langle a, b\rangle$. Then $U(L)$ is generated by a and b, as an associative algebra. But $U(L)$ is the free associative algebra on 2 generators, c and d, say. Define a map $U(L) \to U(L)$ by $c \mapsto a$ and $d \mapsto b$. Since this map is an epimorphism, it is an automorphism (see [4], Proposition 6.8.1, for example). So, $\{a, b\}$ freely generates $U(L)$ and consequently $L = \langle a, b\rangle$ is a free Lie algebra, a contradiction.

We remark that, while L requires 3 generators, it can be deduced from Lemma 8.1 that L_p is a free restricted Lie algebra on 2 generators.

Finally, let L and H be as in Example B. Since L_p is the universal restricted Lie algebra envelope of L, the Lie algebra homomorphism $\Phi : L \to H_p$ extends to a restricted Lie algebra isomorphism $L_p \to H_p$. It now follows from Lemma 8.1 that $U(L)$ and $U(H)$ are isomorphic as Hopf algebras, as was asserted in the Introduction.

9. Lie superalgebras

We now present an example illustrating that the analogous isomorphism problem for enveloping algebras of Lie superalgebras fails utterly.

Let $\mathbb{F}$ be a field of characteristic not 2. In the case of characteristic 3, we add the axiom $[x, x, x] = 0$ in order for the universal enveloping algebra, $U(L)$, of a Lie superalgebra L to be well-defined.

Example D. *Let $L = \mathbb{F}x_0$ be the free Lie superalgebra on one generator x_0 of even degree, and let $H = \mathbb{F}x_1 + \mathbb{F}y_0$ be the free Lie superalgebra on one* *generator x_1 of odd degree, where $y_0 = [x_1, x_1]$. Then $U(L)$ is isomorphic to the polynomial algebra $\mathbb{F}[x_0]$ in the indeterminate x_0. On the other hand, $U(H) \cong \mathbb{F}[x_1, y_0]/I$, where I is the ideal of the polynomial algebra $\mathbb{F}[x_1, y_0]$ generated by $y_0 - 2x_1^2$. Hence, $U(H) \cong \mathbb{F}[x_1] \cong \mathbb{F}[x_0] \cong U(L)$. However, $L \ncong H$ since they do not even have the same dimension.*

10. An application to special Lie PI-algebras

A Lie algebra L is said to be *special* if L can be embedded into an associative algebra A satisfying a polynomial identity. A result of Bahturin (see [1]) asserts that if L is nilpotent-by-abelian then L is special. As an application of Proposition 6.1, we offer another proof of this fact.

Proposition 10.1. *Let L be a Lie algebra such that L' is nilpotent of class c. Then L has an associative envelope satisfying the polynomial identity*

$$[x_1, y_1] \cdots [x_c, y_c] z_{c+1} = 0.$$

Proof. According to Proposition 6.1, $L \cap (L')^c \omega(L) = \gamma_{c+1}(L') = 0$. Consequently, the natural Lie homomorphism $L \to \omega(L)/[\omega(L), \omega(L)]^c \omega(L)$ is an embedding. □

References

[1] Yu.A. Bahturin, *Identical Relations in Lie Algebras* (VNU Science Press, b.v., Utrecht, 1987).

[2] Yu.A. Bahturin, A. Mikhalev, V. Petrogradsky, M. Zaicev, *Infinite-Dimensional Lie Superalgebras*, de Gruyter Exp. Math. **7** (de Gruyter, Berlin, 1992).

[3] J. Chun, T. Kajiwara, J. Lee, Isomorphism theorem on low dimensional Lie algebras, *Pacific J. Math.* **214** (2004), no. 1, 17–21.

[4] P.M. Cohn, *Free rings and their relations* (Second edition, Academic Press, 1985).

[5] M. Hertweck, A counterexample to the isomorphism problem for integral group rings, *Ann. of Math. (2)* **154** (2001), no. 1, 115–138.

[6] M.A. Knus, On the enveloping algebra and the descending central series of a Lie algebra *J. Algebra* **12** (1969), 335–338.

[7] M.I. Kuznetsov, Truncated induced modules over transitive Lie algebras of characteristic p, *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), no. 3, 557–589.

[8] P. Malcolmson, Enveloping algebras of simple three-dimensional Lie algebras, *J. Algebra* **146** (1992), 210-218.

[9] A.A. Mikhalev, A.A. Zolotykh, *Combinatorial aspects of Lie superalgebras* (CRC Press, Boca Raton, FL, 1995).

[10] I.B.S. Passi, S.K. Sehgal, Isomorphism of modular group algebras, *Math. Z.* **129** (1972), 65–73.

[11] D.M. Riley, The dimension subalgebra problem for enveloping algebras of Lie superalgebras, *Proc. Amer. Math. Soc.* **123** (1995), no. 10, 2975–2980.

[12] K. Roggenkamp, L. Scott, Isomorphisms of p-adic group rings, *Ann. of Math. (2)* **126** (1987), no. 3, 593–647.

[13] H. Strade, R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Monographs and Textbooks in Pure and Applied Mathematics **116** (Dekker, New York, 1988).

[14] A.I. Valitskas, A representation of finite-dimensional Lie algebras in radical rings, *Dol. Akad. Nauk SSSR* **279** (1984), no. 6, 1297-1300.

[15] A. Weiss, Rigidity of p-adic p-torsion, *Ann. of Math. (2)* **127** (1988), no. 2, 317–332.

Department of Mathematics, The University of Western Ontario, London, Ontario, N6A 5B7, Canada
E-mail address: dmriley@uwo.ca

Department of Mathematics, The University of Western Ontario, London, Ontario, N6A 5B7, Canada
E-mail address: husefi@uwo.ca