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LIE ALGEBRAS WITH FINITE GELFAND-KIRILLOV DIMENSION

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Abstract. We prove that every finitely generated Lie algebra containing a nilpotent ideal of class c and finite codimension n has Gelfand-Kirillov dimension at most cn. In particular, finitely generated virtually nilpotent Lie algebras have polynomial growth.

1. Introduction and statement of results

Groups with polynomial growth are very well understood. Indeed, according to the famous result of Milnor, Wolf and Gromov ([4]), a finitely generated group has polynomial growth precisely when it contains a nilpotent normal subgroup of finite index. While the class of associative algebras with polynomial growth has not yet yielded a similar characterisation, Berele ([3]) has provided us with an illuminating sufficient condition: every finitely generated associative algebra satisfying a nontrivial polynomial identity has polynomial growth.

In contrast, the class of Lie algebras with polynomial growth remains much of a mystery. On the one hand, Lichtman and Ufnarovski ([8]) proved that the growth of every finitely generated free soluble Lie algebra of derived length $k \geq 3$ is almost exponential (and therefore not polynomial). So not every finitely generated Lie PI-algebra has polynomial growth; in other words, the most obvious Lie-theoretic analogue of Berele's theorem is false.

On the other hand, it follows from results of Shmelkin that the case of derived length 2 is correctly omitted since every n-generator metabelian Lie algebra has Gelfand-Kirillov dimension at most n (see [11], [7] or [6]). Petrogradsky extended this result in [9] to the class of all finitely generated nilpotent-by-nilpotent Lie algebras (*cf.* the Corollary below). Our present aim is to extend Petrogradsky's result to the class of all finitely generated virtually nilpotent Lie algebras. Recall that an algebra is said to be *virtually* in a class $\mathcal{X}$ if it contains an ideal of finite codimension in the class $\mathcal{X}$.

Theorem. *Let L be a finitely generated Lie algebra over a field F. If L contains a nilpotent ideal I of class c such that $dim_F(L/I) = n < \infty$, then GK-$dim(L) \leq cn$. In particular, every finitely generated virtually nilpotent Lie algebra has polynomial growth.*

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In Section 3, we shall provide an example to show that this estimate is precise. Let $\mathcal{N}_c$ denote the variety of all nilpotent Lie algebras of class at most c.

Corollary ([9])**.** *The free k-generator Lie algebra in the variety $\mathcal{N}_c\mathcal{N}_d$ has GK-dimension at most cn, where n is the dimension of the free k-generated Lie algebra in $\mathcal{N}_d$.*

Unlike in the group-theoretic situation, being virtually nilpotent is not a necessary condition for a finitely generated Lie algebra to have polynomial growth. In fact, there exist Lie algebras of GK-dimension 1 that are not even virtually soluble:

Example A. Let F be a field of characteristic zero and denote by W the positive part of the Witt algebra. In other words, W is the Lie F-algebra with basis $\{e_i | \, i \geq 1\}$ and relations $[e_i, e_j] = (j-i)e_{i+j}$. Direct calculations show that GK-dim$(W) = 1$ and yet W is not virtually soluble.

Example B. Let p be a prime and let F be a field of characteristic p. Then according to a result of Shalev ([10]), there exists a graded Lie F-algebra $L = \bigoplus_{i\geq 1} L_i$ of maximal class that is not virtually soluble. Here having maximal class means that L is generated by L_1 and satisfies $\dim_F(L_1) = 2$ and $\dim_F(L_i) = 1$ for all $i \geq 2$. The GK-dimension of any such algebra is 1.

Other known classes of Lie algebras with polynomial growth include Kac-Moody, Virasoro and Cartan Lie algebras. See [5] for details.

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2. Proof of the Theorem

Lemma 1. *Let L be a finitely generated Lie algebra containing a nilpotent ideal I of finite codimension. Then I is finitely generated as an ideal of L.*

Proof. We use induction on the nilpotence class c of I. In the case $c = 1$, L is finitely generated and virtually abelian; therefore, by a result of Amayo and Stewart (see [1]), every ideal of L is finitely generated. Now suppose I has class $c > 1$. Then, by the induction hypothesis, I/I^c is finitely generated as an ideal of L/I^c; in other words, there exists a finite-dimensional subspace U of I such that

$$I = \sum_{k\geq 0} [U, \underbrace{L, \ldots, L}_{k}] + I^c.$$

Because $I^{c+1} = 0$, it follows that I^c is contained in the ideal of L generated by U. Thus I is generated by U, as required. $\square$

Henceforth, L is a fixed finitely generated Lie algebra containing a nilpotent ideal I of class c and finite codimension n in L. Let V be a vector space complement of I in L with basis $\{e_1, \ldots, e_n\}$. For each pair i, j, write $[e_i, e_j] = x_{ij} + v_{ij}$, where $x_{ij} \in I$ and $v_{ij} \in V$. By the previous lemma, there exists a finite generating set $\{x_1, \ldots, x_m\}$ of I that contains each of the x_{ij}. Let U be the subspace of I spanned by $\{x_1, \ldots, x_m\}$ and set $W = U + V$.

Lemma 2. *The subspace W generates L as a Lie algebra.*

Proof. Let S be the subalgebra generated by W. Since $L = I + V$, it suffices to show that $I \subseteq S$. Observe then that

$$I = \sum_{k \geq 0} [U, \underbrace{L, \ldots, L}_{k}] = \sum_{k \geq 0} [U, \underbrace{V, \ldots, V}_{k}] + I^2 \subseteq S \cap I + I^2.$$

It follows that $I^r \subseteq S \cap I + I^{r+1}$ for each $r \geq 1$. But $I^{c+1} = 0$ and so the result follows. □

Let $W^{(k)}$ denote the subspace of L spanned by all monomials of degree at most k in the generating set $\{x_1, \ldots, x_m, e_1, \ldots, e_n\}$ of L. We shall call any monomial of the form

$$[x_{i(1)}, e_{j(1,1)}, \ldots, e_{j(1,s_1)}, x_{i(2)}, e_{j(2,1)}, \ldots, e_{j(2,s_2)}, \ldots, x_{i(t)}, e_{j(t,1)}, \ldots, e_{j(t,s_t)}]$$

straight if we have $j(r,1) \leq \cdots \leq j(r, s_r)$ for each $1 \leq r \leq t$. Finally, we shall denote by I_k the subspace of I spanned by all straight monomials of total degree at most k.

Lemma 3. *$W^{(k)} \subseteq I_k + V$ for each $k \geq 1$.*

Proof. Let μ be any monomial of degree $r \leq k$ in the x_i's and e_j's. We shall use induction on r to prove $\mu \in I_k + V$. The case $r = 1$ is trivial. Suppose then that $r > 1$ and that μ is not straight. Then μ begins (on the left) with a monomial of the form $\mu_1 = [\nu, e_b, e_a]$ where $a < b$. By the Jacobi identity we see that

$$\mu_1 = [\nu, e_a, e_b] + [e_a, e_b, \nu] = [\nu, e_a, e_b] - [\nu, x_{ab} + v_{ab}].$$

But x_{ab} is one of the x_i's, and v_{ab} is a linear combination of the e_j's. Hence, μ_1 is congruent to a 'straighter' monomial modulo a linear combination of monomials of lesser degree. Appealing to the induction hypothesis, it follows that μ is congruent modulo $I_k + V$ to a straighter monomial of the same degree r. Because any monomial will be straightened after a finite series of such transpositions, the lemma is proved. □

We are now ready to complete the proof of our theorem.

Lemma 4. *The GK-dimension of L is at most cn.*

Proof. A simple counting argument followed by crude estimates shows that the number of straight monomials of degree t in the x_i's and total degree r is

$$m^t \sum_{r_1 + \cdots + r_t = r - t} \prod_{i=1}^{t} \binom{r_i + n - 1}{n - 1} \leq m^t r^{tn-1},$$

and thus

$$\dim(I_k) \leq \sum_{r=1}^{k} \sum_{t=1}^{c} m^t r^{tn-1} \leq c m^c k^{cn}.$$

It now follows from the previous lemma that $\dim(W^{(k)}) \leq c m^c k^{cn} + n$, for each $k \geq 1$. Therefore, GK-dim$(L) = \overline{\lim}_{k \to \infty} \log_k(\dim(W^{(k)})) \leq cn$. □

3. A critical example

To illustrate that our upper bound is best possible, we adapt Shmelkin's construction of a verbal wreath product (see [2], for example). For each finite-dimensional Lie algebra L with basis $\{e_1, \ldots, e_n\}$ consider the free right L-module M with generators $\{x_1, \ldots, x_c\}$. Then

$$M_0 = \{x_i e_1^{a_1} \cdots e_n^{a_n} \mid a_j \geq 0,\ 1 \leq i \leq c\}$$

is a basis for M. Next consider the free Lie algebra N in $\mathcal{N}_c$ on the set of generators M_0. One can make N into a right L-module via $[n_1, n_2]g = [n_1 g, n_2] + [n_1, n_2 g]$, $n_1, n_2 \in N$, $g \in L$. The semidirect product $N + L$ is spanned by $\{e_1, \ldots, e_n\}$ together with the set of monomials of degree at most c in the generating set M_0. Furthermore, monomials

$$[x_1 e_1^{a_1} \cdots e_n^{a_n}, x_2 e_1^{b_1} \cdots e_n^{b_n}, \ldots, x_c e_1^{c_1} \cdots e_n^{c_n}]$$

are linearly independent. The number of such monomials of degree at least $c+1$ and at most k is

$$\sum_{r=c+1}^{k} \binom{r-c+cn-1}{cn-1} \geq \sum_{r=c+1}^{k} \frac{(r-c)^{cn-1}}{(cn-1)!} \geq \frac{(k-c)^{cn}}{(cn)!}.$$

It follows that GK-dim$(N+L) \geq cn$. Consequently, GK-dim$(N+L) = cn$ according to our Theorem.

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