FOX-TYPE PROBLEMS IN ENVELOPING ALGEBRAS

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Abstract. Let L be a Lie algebra with the universal enveloping algebra $U(L)$. The augmentation ideal $\omega(L)$ of $U(L)$ is the associative ideal of $U(L)$ generated by L. Let S be a subalgebra of L and n a positive integer. In this paper we prove that $L \cap \omega^2(L)\omega^n(S) = \gamma_{n+2}(S) + \gamma_{n+1}(S \cap \gamma_2(L))$, where $\gamma_{n+2}(S)$ is the $(n+2)^{\text{th}}$ term of the lower central series of S.

1. Introduction

Let L be a Lie algebra and S a subalgebra of L over a field $\mathbb{F}$. We denote the universal enveloping algebra of L by $U(L)$. The augmentation ideal $\omega(L)$ of $U(L)$ is the kernel of the algebra map $\varepsilon : U(L) \to \mathbb{F}$ induced by $\varepsilon(x) = 0$ for every $x \in L$. Thus, $\omega(L) = LU(L) = U(L)L$. For every positive integer n, we denote by $\omega^n(L)$ the n^{th} power of $\omega(L)$ and $\omega^0(L)$ is $U(L)$.

It is proved in [8] and [10] that $L \cap \omega^n(L) = \gamma_n(L)$, the n^{th} term of the lower central series of L. The identification of the subalgebras $L \cap \omega^n(L)\omega^m(S)$ naturally arises in the context of enveloping algebras. The motivation for this sort of problem also comes from its group ring counterpart. Let F be a free group, R a normal subgroup of F, and denote by $\mathfrak{r}$ the kernel of the natural homomorphism $\mathbb{Z}F \to \mathbb{Z}(F/R)$. Recall that the augmentation ideal $\mathfrak{f}$ of the integral group ring $\mathbb{Z}F$ is the kernel of the map $\mathbb{Z}F \to \mathbb{Z}$ induced by $g \mapsto 1$ for every $g \in F$. Fox introduced in [2] the problem of identifying the subgroup $F \cap (1 + \mathfrak{f}^n\mathfrak{r})$ in terms of R. Following Gupta's initial work on Fox's problem ([3]), Hurley ([6]) and Yunus ([14]) independently gave a complete solution to this problem. Hurley developed a theory of basic commutators for free groups which was first initiated by P. Hall ([5]) and Yunus used Lie theoretic techniques and certain representation of integral group rings of free groups due to Gupta and Passi ([4]).

At the same time Yunus considered the Fox problem for free Lie algebras. Let $\mathcal{L}$ be a free Lie algebra and $\mathcal{S}$ an ideal of $\mathcal{L}$. Yunus in [15] identified the subalgebra $\mathcal{L} \cap \omega(\mathcal{S})\omega^n(\mathcal{L})$ in terms of $\mathcal{S}$. We considered the Fox problem for free restricted Lie algebras in [12] and gave a complete solution to the problem.

2000 *Mathematics Subject Classification.* Primary 17B35; Secondary 20C07.

The research is supported by a postdoctoral fellowship from the Department of Mathematics at the University of British Columbia.

More recently, Hurley and Sehgal ([7]) proved that

$$F \cap (1 + \omega^2(F)\omega^n(R)) = \gamma_{n+2}(R)\gamma_{n+1}(R \cap \gamma_2(F)),$$

for every positive integer n. Here, $\gamma_n(R)$ denotes the n^{th} term of the lower central series of R. Our goal in this paper is to provide a similar answer for the analogues problem for Lie algebras. Our main result is as follows:

Main Theorem. *Let L be a Lie algebra over a field. For every subalgebra S of L and every positive integer n, we have*

$$L \cap \omega^2(L)\omega^n(S) = \gamma_{n+2}(S) + \gamma_{n+1}(S \cap \gamma_2(L)).$$

The definitions and some preliminary results are presented in Section 2. The Main Theorem is proved in Section 3. Our techniques in this paper are quite general and can be applied to other categories including restricted Lie algebras, Lie superalgebras, and finite p-groups. We shall mention the analogues results for restricted Lie algebra in Section 4. The results for p-groups will appear in [13].

2. Definitions and preliminary results

Let L be a Lie algebra with an ordered basis X over a field $\mathbb{F}$. Lower central series of L are defined by setting $\gamma_1(L) = L$ and $\gamma_n(L) = [\gamma_{n-1}(L), L]$, for every $n \geq 2$. Let S be a subalgebra of L and $y \in S$. The *height*, $\nu_S(y)$, of y in S is defined to be the largest subscript n such that $y \in \gamma_n(S)$ if n exists and to be infinite if it does not. If we consider y as an element of L then we denote the height of y by $\nu(y)$. A basis Y of S is said to be ordered with respect to the lower central series of S provided that $y_1 \leq y_2$ implies that $\nu_S(y_1) \leq \nu_S(y_2)$, for every $y_1, y_2 \in Y$.

The Poincaré-Birkhoff-Witt (PBW) theorem (see [1], for example) allows us to view L as a Lie subalgebra of $U(L)$ in such a way that $U(L)$ has a basis consisting of PBW monomials of the form $x_1^{a_1} \cdots x_t^{a_t}$ where $x_1 \leq \cdots \leq x_t$ in X and t and each a_i are non-negative integers. *Degree* of a PBW monomial $x_1^{a_1} \cdots x_t^{a_t}$ is defined to be $\sum_{i=1}^{t} a_i$ and *weight* of any such monomial is $\sum_{i=1}^{t} a_i\nu(x_i)$.

The graded Lie algebra of L corresponding to its lower central series is given by $\mathrm{gr}(L) = \oplus_{i \geq 1}\gamma_i(L)/\gamma_{i+1}(L)$. Let $X = \{\bar{x}_i\}_{i \in \mathcal{I}}$ be a homogeneous basis of $\mathrm{gr}(L)$. Take a coset representative x_i for each $\bar{x}_i$.

Lemma 2.1. *Let L be a Lie algebra. Then the set of all PBW monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_s}^{a_s}$ with the property that $\sum_{k=1}^{s} a_k\nu(x_{i_k}) = n$ forms an $\mathbb{F}$-basis for $\omega^n(L)$ modulo $\omega^{n+1}(L)$, for every $n \geq 1$.*

We need to fix some notations for the next lemma. Let n be a sufficiently large integer. Let $\bar{X}_n$ be a homogeneous basis for $\bigoplus_{i=1}^{n} \gamma_i(L)/\gamma_{i+1}(L)$ and extend it to a homogeneous basis $\bar{X}$ of $\mathrm{gr}(L)$. Let X_n and X, $X_n \subseteq X$, be fixed sets of representatives of $\bar{X}_n$ and $\bar{X}$. Since every $x \in X \setminus X_n$ lies in $\gamma_{n+1}(L)$, we can extend X to a basis $X \cup Y$ of L by choosing a complement basis Y of $X \setminus X_n$ in $\gamma_{n+1}(L)$. Finally we order $X \cup Y$ is some way. With these notations we have:

Lemma 2.2. *For every positive integer $m \leq n+1$, $\omega^m(L)$ is spanned by all PBW monomials in X of weight at least m.*

Proof. Let V_t be the vector space spanned by all products $y_{k_1} \cdots y_{k_t}$, where $y_{k_1}, \ldots, y_{k_t} \in Y$. Clearly the assertion is true for all elements in $V_1 \cap \omega^m(L) = L \cap \omega^m(L) = \gamma_m(L)$. Suppose, by induction, that the assertion is true for all elements in $V_t \cap \omega^m(L)$ and let $u \in V_{t+1} \cap \omega^m(L)$. If u is not a PBW monomial, we may assume that $u = vy_sy_rw$, where $y_s > y_r$ and $v \in V_\ell$, $w \in V_{k-\ell-1}$. We have $u = vy_ry_sw + v[y_s, y_r]w$. We now observe that $[y_s, y_r]$ is a linear combination of basis elements $y_k \in Y$ such that $\nu(y_k) \geq \nu([y_s, y_r])$ (or $\nu(y_k) \geq n+1$ if $[y_s, y_r] \in \gamma_{n+1}(L)$). Thus, $u = vy_ry_sw$ modulo $V_t \cap \omega^m(L)$. Hence, modulo $V_t \cap \omega^m(L)$, u can be transformed to a PBW monomial of weight m by finitely many such transpositions. The assertion then follows from the induction hypothesis. □

Note that the above lemma may replace Proposition 3.1 in [8], see [10] for more details.

Proposition 2.3. *Let L be a Lie algebra and S a Lie subalgebra of L. Then for every positive integer n, we have*

$$\omega(S) \cap \omega^2(L)\omega^n(S) = \omega^{n+2}(S) + \omega(S \cap \gamma_2(L))\omega^n(S).$$

Proof. Let $R = S \cap \gamma_2(L)$. Clearly, $\omega^{n+2}(S)$ and $\omega(R)\omega^n(S)$ are both contained in $\omega^2(L)\omega^n(S)$. Now let $w \in \omega(S) \cap \omega^2(L)\omega^n(S)$. Let $\bar{X}_1$ be a basis for S/R. Since S/R embeds into $L/\gamma_2(L)$, we can extend $\bar{X}_1$ to a basis $\bar{Y}_1$ of $L/\gamma_2(L)$. Let X_1 and Y_1, $X_1 \subseteq Y_1$, be fixed sets of representatives of $\bar{X}_1$ and $\bar{Y}_1$, respectively. We can extend X_1 to a basis X of S by choosing a basis for R. Clearly, X and Y_1 are linearly independent and so we can extend $X \cup Y_1$ to a basis Y of L. We order Y in a way that every $x \in X$ is greater than every $y \in Y \setminus X$. Since $w \in \omega^2(L)\omega^n(S)$, w is a sum of elements uv, where each u is a PBW monomial of weight at least two in the basis Y, by Lemma 2.2, and $v \in \omega^n(S)$ is a sum of PBW monomials in the basis X. Note that if u starts with an x, $x \in X_1$, then uv lies in $\omega^{n+2}(S)$ because u has weight two. Also if u starts with an x, $x \in X \setminus X_1$, then uv lies in $\omega(R)\omega^n(S)$. So,

$$w = \sum uv \text{ modulo } \omega^{n+2}(S) + \omega(R)\omega^n(S),$$

where each u starts with a $y \in Y \setminus X$. But $w \in \omega(S)$ and so w can be written as a linear combination of PBW monomials in the basis X only. It now follows from the independence of PBW monomials that

$$\sum uv = 0,$$

and thus $w \in \omega^{n+2}(S) + \omega(R)\omega^n(S)$, as required. □

We shall also need the following result from [10].

Lemma 2.4. *Let S be any subalgebra of a Lie algebra L. Then for every positive integer n, we have*

$$L\cap\omega(L)\omega^n(S)=\gamma_{n+1}(S).$$

3. Proof of the Main Theorem

Let R be a Lie subalgebra of a Lie algebra S and n a positive integer. We fix the following notation for this section. For every $i\geq 1$, consider the natural sequence of vector spaces:

$$\frac{\gamma_i(R)\cap\gamma_{i+1}(S)}{\gamma_{i+1}(R)}\xhookrightarrow{f_i}\frac{\gamma_i(R)}{\gamma_{i+1}(R)}\xrightarrow{g_i}\frac{\gamma_i(R)}{\gamma_i(R)\cap\gamma_{i+1}(S)}\xhookrightarrow{h_i}\frac{\gamma_i(S)}{\gamma_{i+1}(S)}$$

Note that the maps g_i are surjective. There is an induced natural sequence of graded vector spaces as follows:

$$\begin{tikzcd}
\mathcal{V}=\bigoplus_{i=1}^n \frac{\gamma_i(R)\cap\gamma_{i+1}(S)}{\gamma_{i+1}(R)} \arrow[r, hook, "f"] & \mathcal{R}=\bigoplus_{i=1}^n \frac{\gamma_i(R)}{\gamma_{i+1}(R)} \arrow[d, two heads, "g"] & \\
& \mathcal{W}=\bigoplus_{i=1}^n \frac{\gamma_i(R)}{\gamma_i(R)\cap\gamma_{i+1}(S)} \arrow[r, hook, "h"] & \mathcal{S}=\bigoplus_{i=1}^n \frac{\gamma_i(S)}{\gamma_{i+1}(S)}
\end{tikzcd}$$

Since g is surjective, we can choose a linearly independent homogeneous subset $\bar{X}_n\subseteq\mathcal{R}$ so that $g(\bar{X}_n)$ is a homogeneous basis for $\mathcal{W}$. Let $\bar{Y}_n$ be a homogeneous basis for $\mathcal{V}$. Clearly, $\bar{X}_n\cup\bar{Y}_n$ is a homogeneous basis of $\mathcal{R}$. Since h is injective, we can extend $g(\bar{X}_n)$ to a basis $g(\bar{X}_n)\cup\bar{Z}_n$ of $\mathcal{S}$. Let X_n, Y_n, Z_n be fixed sets of representatives of $\bar{X}_n, \bar{Y}_n, \bar{Z}_n$, respectively. Note that $X_n\cup Y_n$ is linearly independent modulo $\gamma_{n+1}(R)$. So, $X_n\cup Y_n$ can be extended to a basis of R by choosing a basis X'_n of $\gamma_{n+1}(R)$. Let $X=X_n\cup X'_n$. Thus $X\cup Y_n$ is a basis of R. Observe that X and Z_n are linearly independent and so $X\cup Z_n$ can be extended to a basis Z of S.

We order X with respect to the lower central series of R. We then order $X\cup Y_n$ assuming that every $x\in X$ is less than every $y\in Y_n$. We also order the basis Z of S so that every $x\in X$ is less than every $z\in Z\setminus X$. Let E_2 be the vector space spanned by the PBW monomials in Z of degree at least two.

Lemma 3.1. *Let S be Lie algebra and R a Lie subalgebra of S. Then for every positive integer n and non-negative integer k with $k\leq n-1$, we have*

$$\omega^{n-k}(R)\omega^k(S)\subseteq\omega^{n+1}(S)+E_2+\omega^n(R).$$

Proof. We use induction on k; the case $k=0$ being trivial. Let $w\in\omega^{n-k-1}(R)\omega^{k+1}(S)$. By Lemma 2.2, w is a sum of elements uv, where each u lies in $\omega^{n-k-1}(R)$ and is a PBW monomial in the basis $X\cup Y_n$ and each v lies in $\omega^{k+1}(S)$ and is a PBW monomial in the basis Z. Note that if u involves a basis element $y\in Y_n$ or $x\in X'_n$ then uv lies in $\omega^{n+1}(S)$. So, modulo $\omega^{n+1}(S)$, we can assume each u is a PBW monomial in X_n. Now suppose that $v=zv_1$, where $z\in Z$. If $z\notin X_n$ then $z\in X'_n$ or otherwise $z\in Z\setminus X$

in which cases $uv \in E_2$. Thus, modulo $\omega^{n+1}(S) + E_2$, we can assume that each $u \in \omega^{n-k-1}(R)$ is a PBW monomial in X_n and each $v \in \omega^{k+1}(S)$ is of the form $v = zv_1$, where $z \in X_n$. Note that $z \in X_n$ means that $z \in \gamma_i(S)$ if and only if $z \in \gamma_i(R)$. Thus,

$$uv = (uz)v_1 \in \omega^{n+i-k-1}(R)\omega^{k+1-i}(S) \subseteq \omega^{n-k}(R)\omega^k(S).$$

Hence, $w \in \omega^{n+1}(L) + E_2 + \omega^{n-k}(R)\omega^k(S)$. The result now follows by the induction hypothesis. □

Proposition 3.2. *Let R be a Lie subalgebra of a Lie algebra S. Then for every positive integer n, we have*

$$S \cap (\omega^{n+1}(S) + \omega(R)\omega^{n-1}(S)) \subseteq \gamma_{n+1}(S) + \gamma_n(R).$$

Proof. By the previous lemma, it is enough to prove that

$$S \cap (\omega^{n+1}(S) + \omega^n(R) + E_2) \subseteq \gamma_{n+1}(S) + \gamma_n(R).$$

Let $w \in S \cap (\omega^{n+1}(S) + \omega^n(R) + E_2)$. Thus, modulo $\omega^{n+1}(S) + E_2$, we can write w as a linear combination of PBW monomials $u \in \omega^n(R)$ in the basis $X \cup Y_n$, by Lemma 2.2. But if u involves a basis element $y \in Y_n$ or $x \in X'_n$, then u lies in $\omega^{n+1}(S)$. So, modulo $\omega^{n+1}(S) + E_2$, each u is a PBW monomial in X_n that lies in $\omega^n(R)$. But if degree of u is more than one then $u \in E_2$. It now follows that

$$w = \sum \alpha_i x_i, \text{ modulo } \omega^{n+1}(S) + E_2,$$

where each x_i lies in $X_n \cap \omega^n(R)$, each $\alpha_i \in \mathbb{F}$ and almost all α_i's are zero. Hence, $w - \sum \alpha_i x_i \in S \cap (\omega^{n+1}(S) + E_2)$. But S is linearly independent with E_2 by the PBW Theorem. It follows that $w - \sum \alpha_i x_i \in S \cap \omega^{n+1}(S) = \gamma_{n+1}(S)$. So, $w \in \gamma_n(R) + \gamma_{n+1}(S)$, as required. □

The Main Theorem is the equality of (1) and (3) in the following theorem.

Theorem 3.3. *Let L be a Lie algebra and S a Lie subalgebra L. For every positive integer n, the following subalgebras of L coincide.*

1. $\gamma_{n+2}(S) + \gamma_{n+1}(S \cap \gamma_2(L))$,
2. $L \cap (\omega^{n+2}(S) + \omega(S \cap \gamma_2(L))\omega^n(S))$,
3. $L \cap \omega^2(L)\omega^n(S)$.

Proof. Let $R = S \cap \gamma_2(L)$. Certainly, $\gamma_{n+1}(R) \subseteq \omega(R)\omega^n(S)$ and thus

$$\gamma_{n+2}(S) + \gamma_{n+1}(S \cap \gamma_2(L)) \subseteq L \cap (\omega^{n+2}(S) + \omega(S \cap \gamma_2(L))\omega^n(S)).$$

Note that, by Lemma 2.4,

$$L \cap (\omega^{n+2}(S) + \omega(R)\omega^n(S)) \subseteq L \cap \omega(L)\omega^n(S) = \gamma_{n+1}(S).$$

Thus, by Proposition 3.2, we have

$$L \cap (\omega^{n+2}(S) + \omega(R)\omega^n(S)) \subseteq S \cap (\omega^{n+2}(S) + \omega(R)\omega^n(S)) \subseteq \gamma_{n+2}(S) + \gamma_{n+1}(R).$$

Also, $L\cap\omega^2(L)\omega^n(S)\subseteq L\cap\omega(L)\omega^n(S)=\gamma_{n+1}(S)$, by Lemma 2.4. Thus, by Proposition 2.3, we have

$$L\cap\omega^2(L)\omega^n(S)=L\cap\omega(S)\cap\omega^2(L)\omega^n(S)=L\cap(\omega^{n+2}(S)+\omega^n(S)\omega(R)).$$

The proof is complete. $\square$

4. Restricted Lie algebras

Recall that a restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p>0$ is a Lie algebra that further possesses a unary $[p]$-map with the properties modeled on exponentiation by p in an associative algebra.

Let L be a restricted Lie algebra with a p-map $^{[p]}$. We denote the restricted enveloping algebra of L by $u(L)$. The augmentation ideal, $\omega(L)$, of $u(L)$ is the associative ideal of $u(L)$ generated by L. The dimension subalgebras of L are defined by

$$D_n(L)=L\cap\omega^n(L)=\sum_{ip^j\geq n}\gamma_i(L)^{[p]^j},$$

for every $n\geq 1$. Here, $\gamma_i(L)^{[p]^j}$ is the restricted Lie subalgebra of L generated by all x^{p^j}, $x\in\gamma_i(L)$. It is known that $[D_n(L),D_m(L)]\subseteq\gamma_{m+n}(L)$ and $D_n(L)^{[p]}\subseteq D_{np}(L)$, for every $m,n\geq 1$. For an account about the dimension subalgebras see [9] or [11].

Note that the role of lower central series in Lie algebras is played by the dimension subalgebras in restricted Lie algebras. With this observation in mind, the corresponding results for restricted Lie algebras can be established along the same lines.

Proposition 4.1. *Let L be a restricted Lie algebra and S a restricted Lie subalgebra of L. Then for every positive integer n, we have*

$$\omega(S)\cap\omega^2(L)\omega^n(S)=\omega^{n+2}(S)+\omega(S\cap D_2(L))\omega^n(S).$$

Theorem 4.2. *Let L be a restricted Lie algebra and S a restricted Lie subalgebra of L. For every positive integer n, the following subalgebras of L coincide.*

(1) $D_{n+2}(S)+D_{n+1}(S\cap D_2(L))$,
(2) $L\cap(\omega^{n+2}(S)+\omega(S\cap D_2(L))\omega^n(S))$,
(3) $L\cap\omega^2(L)\omega^n(S)$.

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