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Application of Pade Approximation to Problems of Fluid Dynamics

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Abstract

It is well known that nonlinear sequence transforms are very effective accelerators of convergence on monotone and alternating sequences. A given acceleration method refines its approximation procedure by progressively absorbing a greater number of terms of a sequence in the transform it employs. In this work the necessary mathematical formulation for calculation of Pade Approximant is performed and routine for its numerical evaluation is developed. The approximation is then applied to two concrete problems of fluid dynamics and merits and efficiency of this approach relative to other approaches to the solution of the problem is discussed.

Introduction

There are many methods for accelerating the convergence of sequences and the subsequent evaluation of the limit of an infinite sequence [1,2,3]. These methods generally employ specific sequence-to-sequence transformations and belong, accordingly, to two broad classes: linear and non-linear. In a comparative study of a number of these methods, Smith and Ford [2] have concluded that nonlinear methods are more general in scope than the linear ones. The nonlinear methods they have reviewed are all generalizations of Aitken's Δ^2 [10]. A particular sequence transform requires a finite number of terms of the sequence on which it is applied. This number is, therefore, a parameter of the transform. A given acceleration method refines its approximation procedure by progressively absorbing a greater number of terms of a sequence in the transform it employs. The number of significant digits in the final value increases, correspondingly, to a limit imposed by round-off errors and/or the effectiveness of the method. If the number of digits to which the evaluation of the limit is accurate in a given method decreases finally as more terms of a sequence are used, we call the method unstable. In all methods the accuracy of evaluation is usually checked against a known result or by noting the consistent appearance of a certain number of digits. It would be more convenient if there existed an independent estimate of the error at each point of the calculation.

A sequence transform uses a finite number of terms of one sequence to generate each term of an auxiliary sequence. Such a primary sequence may be a sequence of numbers or a sequence of functions, an example of the latter being the partial sums of a power series. The most widely used nonlinear sequence transforms are Aitken's Δ^2–transform, Shanks's e-transform, Wynn's ε-transform and Levin's u-transform. A unified discussion of these transforms is found in ref. [5]. If the limit of the generated sequence is the same as that of the original sequence, the sequence transform is said to be regular. It is well known that nonlinear sequence transforms are very effective accelerators of convergence on monotone and alternating sequence of numbers. Interestingly, they induce convergence in divergent sequences and hence are valid methods of summation. When a nonlinear sequence transform is applied to the sequence of partial sums of a power series, it generates approximants in the form of rational functions. The representation of functions by rational approximants has been a major field of endeavor, especially for functions represented by divergent series expansions. Uses of the rational function representation of a function whose series expansion is known are too numerous to mention.

The Pade approximant has been used most frequently in tackling divergent series encountered in theoretical physics [6], although the methods of Euler and Borel have also been used to some extent. It is well known that the nonlinear sequence transform ε is closely related to the Pade approximant. The superiority of the u-transform over the ε-transform in summing a wide class of convergent and divergent test sequences of numbers, both real and complex[5,7], lends encouragement to the conjecture that the former may also prove useful as a generator of rational approximants, at least for a certain class of power series. A recent comparison between the two methods made on a divergent perturbation series expansion for the excluded volume effect in the theory of polymer solutions extends support to this surmise [8].

In realistic perturbation problems only a few terms of a perturbation series can be calculated before a state of exhaustion is reached. Therefore a summation algorithm is needed which requires as input only a finite number of terms of a divergent series. Then as each new term is computed, it is immediately folded in with the others to give a new and improved estimate of the exact sum of the divergent series. A well-known summation method having this property is called Pade summation. In this work we focus our attention only to Pade approximation.

Pade Approximation

The idea of Pade summation is to replace a power series $\sum_{n=0}^{\infty} a_n x^n$ by a sequence of rational functions (a rational function is a ratio of two polynomials) of the form

$$P_M^N(x) = \frac{\sum_{n=0}^{N} A_n x^n}{\sum_{n=0}^{M} B_n x^n} \tag{1}$$

where we choose $B_0=1$ without loss of generality. We choose the remaining *(M+N+1)* coefficients $A_0, A_1, \ldots, A_N$, $B_1, B_2, \ldots, B_M$, so that the first *(M+N+1)* terms in the Taylor series expansion of $P_M^N(x)$ match the first *(M+N+1)* terms of the power series $\sum_{n=0}^{\infty} a_n x^n$. The resulting rational function $P_M^N(x)$ is called a Pade Approximant.

We will see that constructing $P_M^N(x)$ is very useful. If $\sum_{n=0}^{\infty} a_n x^n$ is a power series representation of the function *f(x)*, then in many instances $P_M^N(x) \longrightarrow f(x)$ as $N, M \longrightarrow \infty$, even if $\sum_{n=0}^{\infty} a_n x^n$ is a divergent series. Usually we consider only the convergence of the Pade sequences $P_0^j, P_1^{1+j}, P_2^{2+j}, P_3^{3+j}, \ldots$ having $N=M+J$ with J fixed and $M \longrightarrow \infty$. The special sequence J=0 is called the *diagonal* sequence.

The full power series representation of a function need not be known to construct a Pade approximant-we need just the first *M+N+1* terms. Since Pade approximants involve only algebraic operations, they are more convenient for computational purposes than Borel summation, which requires one to integrate over an infinite range the analytic continuation of a function defined by a power series. In fact, the general Pade approximant can be expressed in terms of determinants.

The Pade approximant $P_M^N(x)$ is determined by a simple sequence of matrix operations. The coefficients $B_1, \ldots, B_M$ in the denominator may be computed by solving the matrix Equation

$$A\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_M \end{bmatrix} = -\begin{bmatrix} a_{N+1} \\ a_{N+2} \\ \vdots \\ a_{N+M} \end{bmatrix}, \qquad (2)$$

where a is an M×M matrix with entries $a_{ij} = a_{N+i-j}$ ($1 \leq i,j \leq M$). Then the coefficients $A_0, A_1, \ldots, A_N$ in the numerator are given by

$$A_n = \sum_{j=0}^{n} a_{n-j} B_j, \quad 0 \leq n \leq N \qquad (3)$$

where $B_j=0$ for $j > M$. Equations (2) and (3) are derived by equating coefficients of $1, x, \ldots, x^{N+M}$

$$\sum_{j=0}^{N+M} a_j x^j \sum_{k=0}^{M} B_k x^k - \sum_{n=0}^{N} A_n x^n = O(x^{N+M+1}), x \to 0 \qquad (4)$$

which is just a restatement of the definition of Pade approximants.

In order to evaluate the Pade Approximant for a given series numerically, we have developed a standard routine namely PADE (K, COEF, X, PDAPX) where

K is an input integer which indicates number of the coefficient which is obtained from the given series,
COEF is an input which represents the sequential terms of given series which is taken from the CALLing program
X is an input variable of the Taylor series and

PDAPX the rational approximation as an output.

To implement the routine one has to follow the following procedure:

- Calculate the coefficients (COEF) of the Taylor series using the main program.
- Using these terms form an M X M matrix a by the relation aij = COEF(N+i-j) , where 1<i,j< M
- The matrix equation above is solved. For solving this equation we have used another standard SUBROUTINE GAUSSJ(A,N,NP,B,M,MP) where,

A is an input matrix of N by N elements stored in array of physical dimensions NP by NP,
B is an input matrix of N by M containing the M right hand side vectors, stored in array of physical dimensions NP by MP,
and on output,
A is replaced by the matrix inverse, and
B is replaced by the corresponding set of solution vectors.

After computing the coefficients of the denominator one calculates the coefficient of numerator using the coefficient of denominator and the coefficient of the given test series i.e.using the relation shown in the theorem.

The total sum of the numerator is evaluated using the coefficient of numerator and putting the value of x in the given relation.

Subsequently the total sum of denominator is evaluated using the coefficient of denominator and putting the value of x in the defined relation.

Finally, Pade approximants (PDAPX) is obtained by dividing the value of the sum of numerator by the sum of denominator.

Case Studies

Case I

Paul and Hossain [11] have investigated the problem on the unsteady two-dimensional flow of a viscous incompressible fluid past an insulated permeable plate assuming that the free-stream and the transpiration velocity oscillate about constant means. In that analysis small amplitude oscillation has been considered. Using appropriate transformations the governing two-dimensional boundary layer equations of motion and the energy have been reduced to following equations:

Steady parts:

$$f''' + ff'' = 0 \tag{1}$$

$$\frac{1}{\Pr}\theta'' + f\theta' + 2f''^2 = 0 \tag{2}$$

with the boundary conditions

$$f = \gamma, \quad f' = 0, \quad \theta' = 0 \text{ at } \eta = 0$$
$$f' = 0, \quad \theta = 0 \text{ as } \eta \to \infty \tag{3}$$

Unsteady parts:

$$F''' + fF'' + f''F + 2i\xi(1 - F') = 2\xi\left(f'\frac{\partial F'}{\partial \xi} - f''\frac{\partial F}{\partial \xi}\right) \tag{4}$$

$$\frac{1}{\Pr}\phi'' + f\phi' - \theta' F - 2i\xi\phi + 4f''F'' = 2\xi\left(f'\frac{\partial \phi}{\partial \xi} - \theta'\frac{\partial F}{\partial \xi}\right) \tag{5}$$

with boundary conditions

$$F(0,\xi) = \gamma, \quad F'(0,\xi) = 0, \quad \phi'(0,\xi) = 0$$
$$F'(\infty,\xi) = 1, \quad \phi(\infty,\xi) = 0 \tag{6}$$

where,

$$\gamma = \mp S\sqrt{\frac{2}{\nu U_\infty}}, \quad \Pr = \frac{\nu}{\alpha} \tag{7}$$

where the functions f, θ and F, ϕ are, respectively the dimensionless steady and fluctuating stream function and the temperature function, ξ is the local frequency number, U_∞ being the mean free stream velocity.

Now, the unsteady part of this problem has been solved applying the Pade Approximation method and the result has been compared with those obtained by the series solution method and finite difference method. The results are summarized in the following figures:

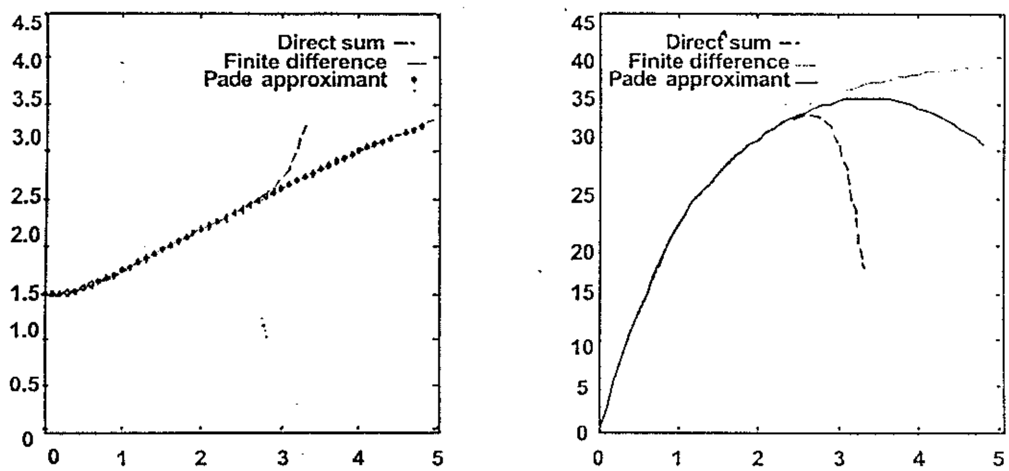

Figure 1: Amplitude and phase of the skin-friction against small ξ while Pr=0.7 & Sc=0.5.

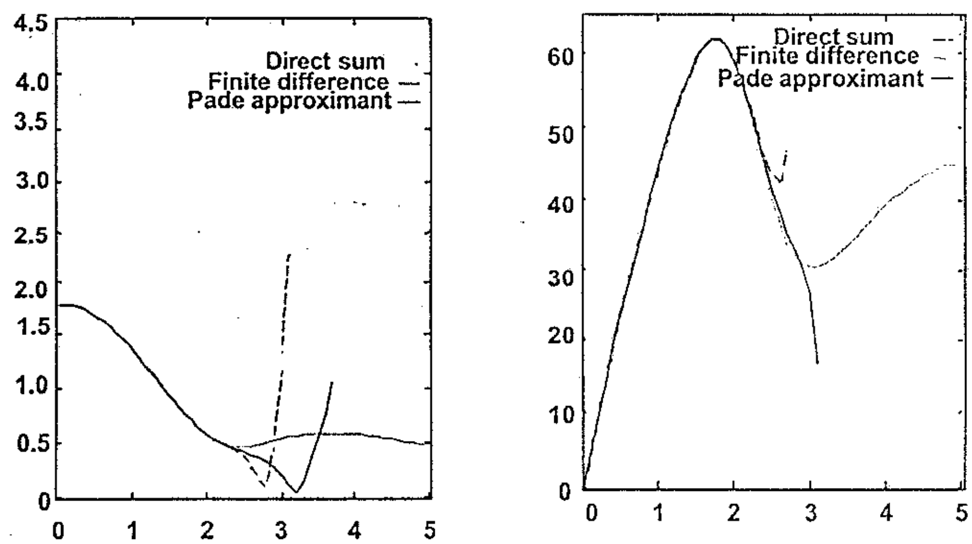

Figure 2: Amplitude and phase of the surface temperature against small ξ with Pr=0.7 & Sc=0.5.

By observing the above figures we can conclude that Pade Approximation method is better than any other methods. Here calculated values are the amplitude and phase angle of the fluctuating skin-friction and fluctuating surface temperature respectively.

Case II:

M. A. Hossain and S. Hossain [12] have investigated the problem on the response of conjugate heat and mass transfer to laminar free convection boundary layer flow of a viscous incompressible fluid a vertical plate considering small amplitude oscillations to surface temperature and surface species concentration to a respective steady non-zero mean. Using appropriate transformations the governing two-dimensional the heat transfer and the mass transfer have been reduced to following equations:

For the steady flow:

$$f''' + \frac{n+3}{4} ff'' - \frac{n+1}{2} f'^2 + (1-w)g + wh = 0 \qquad (8)$$

$$\frac{1}{\Pr} g'' + \frac{n+3}{4} fg' - nf'g = 0 \qquad (9)$$

$$\frac{1}{Sc}h'' + \frac{n+3}{4}fh' - nf'h = 0 \tag{10}$$

$$f(0) = f'(0) = 0, \quad g(0) = 1, \quad h(0) = 1$$
$$f'(\infty) = g(\infty) = h(\infty) = 0 \tag{11}$$

and for the fluctuating flow:

$$F''' + \frac{n+3}{4}(fF'' + f''F) - (n+1)f'F' + (1-w)G + wH - i\xi F' = \frac{(1-n)}{2}\xi\left(f'\frac{\partial F'}{\partial \xi} - f''\frac{\partial F}{\partial \xi}\right) \tag{12}$$

$$\frac{1}{\Pr}G'' + \frac{n+3}{4}(fG' + g'F) - n(f'G + gF') - i\xi G = \frac{(1-n)}{2}\xi\left(f'\frac{\partial G}{\partial \xi} - g'\frac{\partial F}{\partial \xi}\right) \tag{13}$$

$$\frac{1}{Sc}H'' + \frac{n+3}{4}(fH' + h'F) - n(f'H + hF') - i\xi H = \frac{(1-n)}{2}\xi\left(f'\frac{\partial H}{\partial \xi} - h'\frac{\partial F}{\partial \xi}\right) \tag{14}$$

$$F(\xi,0) = F'(\xi,0) = 0, \quad G(\xi,0) = 1, \quad H(\xi,0) = 1$$
$$F'(\xi,\infty) = H(\xi,\infty) = G(\xi,\infty) = 0 \tag{15}$$

where the functions f, g, h and F, G, H are, respectively the dimensionless steady and fluctuating stream function and the temperature function, ξ is the local frequency number, U_∞ is the mean free stream velocity.

Now, as before, the numerical solution of the unsteady part of this problem is found by applying the Pade approximation method and the result is compared with those obtained by series solution and finite difference methods. A comparative picture of the results are shown in the following figures:

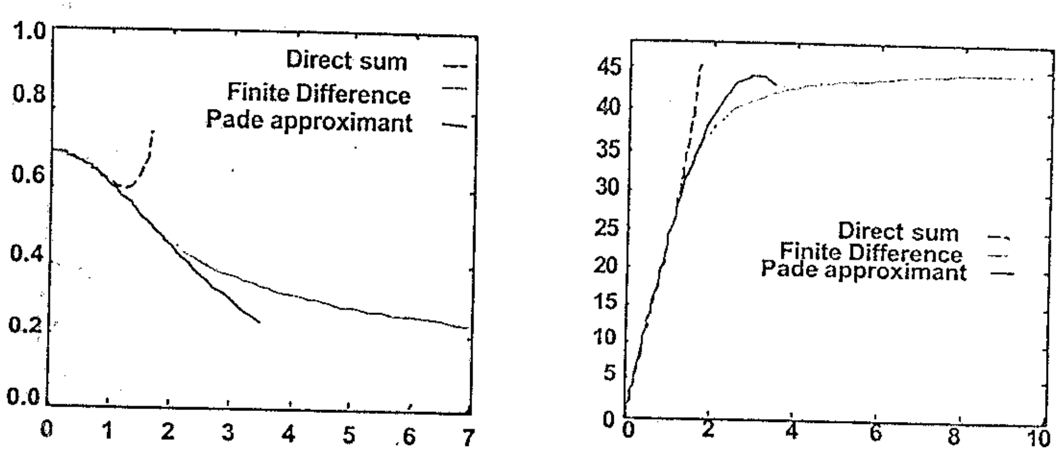

Figure 3: Amplitude and phase of the shear-stress against small ξ while Pr=0.7, Sc=0.22

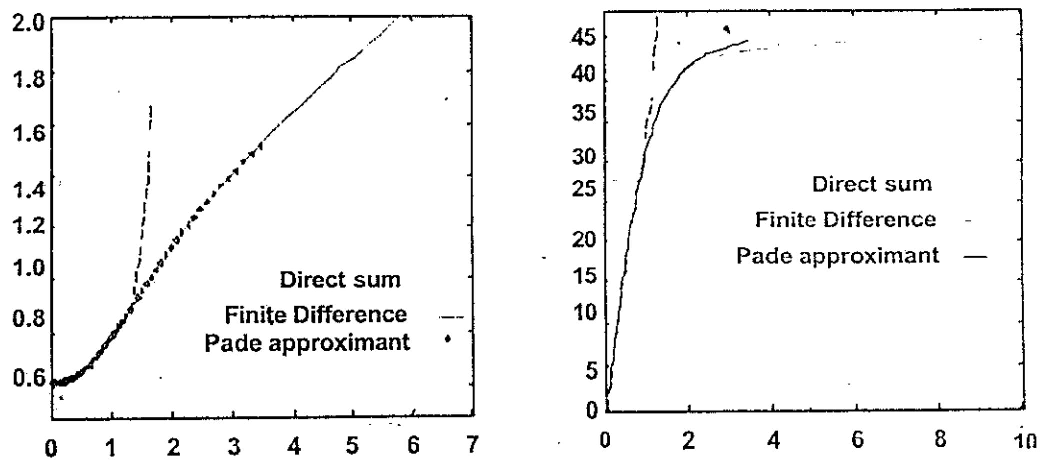

Figure 4: Amplitude and phase of the heat-transfer against small ξ while Pr=0.7, Sc=0.22

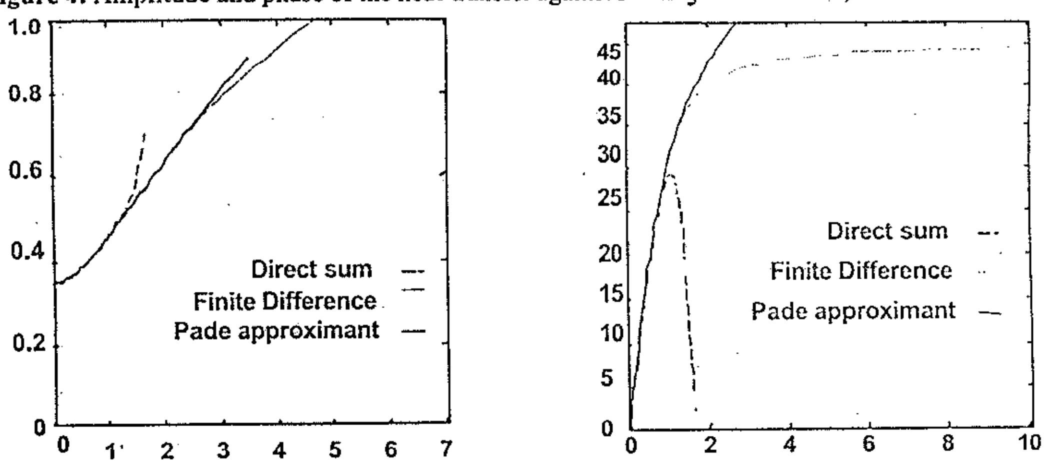

Figure 5: Amplitude and phase of the mass-transfer against small ξ while Pr=0.7, Sc=0.22

By observing the above figures we again conclude that Pade approximation method is better than other methods. Here calculated values are the amplitude and the phase angle of the fluctuating skin-friction and fluctuating surface temperature.

Conclusion

The finding that Pade approximation applied on a power series expansion of a function leads to approximants that are usually better representation of the function, as judged against other solutions of the problems discussed in this work. This is a considerable improvement over standard linearization scheme coupled with a power series expansion. Additional improvements may be obtained by employing new interpolants obtained from other non-linear transforms, as in the case of approximants discussed in ref. [3].

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