Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 9**

Ring Theory

Problem 1: Suppose that R is a ring and that P is a left R-module. We call P projective if the canonical map

$$\pi\colon R^{(P)} \to P,\ (r_p)_{p\in P} \mapsto \sum_{p\in P} r_p p$$

is a split epimorphism, i.e., if there exists an R-linear map $\iota\colon P \to R^{(P)}$ with $\pi \circ \iota = \mathrm{id}_P$. Show that the following conditions are equivalent:

1. P is projective.
2. If F is a free R-module, every epimorphism $\pi\colon F \to P$ splits (in the sense explained above).
3. P is isomorphic to a direct summand of a free module.
4. Every (R-linear) epimorphism $\pi\colon M \to P$ splits.
5. If $f\colon M \to N$ is an R-linear epimorphism, then every R-module homomorphism $g\colon P \to N$ can be lifted to M is the sense that there is an R-module homomorphism $h\colon P \to M$ with $f \circ h = g$.

(Remark: Here $R^{(P)}$ denotes the direct sum of copies of R with itself, which we can write as

$$R^{(P)} = \bigoplus_{p\in P} R = \{f\colon P \to R \mid f(p) \neq 0 \text{ for at most finitely many } p \in P\}$$

In the notation above, we have $f(p) = r_p$.) (25 points)

Problem 2: Suppose that R is a ring and that I is a (left) R-module. We call I injective if every (R-linear) monomorphism $\iota\colon I \to M$ into an arbitrary (left) R-module splits in the sense that there is an R-linear map $\pi\colon M \to I$ with $\pi \circ \iota = \mathrm{id}_I$. Show that the following conditions are equivalent:

1. I is injective.
2. If $f\colon M \to N$ is an R-linear monomorphism, then every R-module homomorphism $g\colon M \to I$ can be extended to N is the sense that there is an R-module homomorphism $h\colon N \to I$ with $h \circ f = g$. (25 points)

Problem 3: Show that the following conditions on a ring R are equivalent:

1. R is semisimple.
2. Every left R-module is projective.
3. Every left R-module is injective. (25 points)

Problem 4: Suppose that R is a principal ideal domain. A (left) module M over R is called divisible if, for every $m \in M$ and every nonzero $r \in R$, there exists $n \in N$ such that $rn = m$. Show that every injective R-module is divisible.

(Remark: A commutative ring is called a principal ideal domain if it does not contain nontrivial zero divisors and if every ideal is principal, i.e., generated by one element. It follows from Baer's criterion that the converse of this statement is also true, but you do not need to show that.) (25 points)

Due date: Tuesday, November 28, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.