Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 8**

Ring Theory

Problem 1: Suppose that M is a left module over the ring R and that N and P are submodules of M. Show that the mapping

$$N/(N \cap P) \to (N+P)/P, \ \bar{m} \mapsto \bar{m}$$

is an isomorphism. (25 points)

(Remark: Note that the bar has two different meanings on both sides of the definition. This theorem is often called the first isomorphism theorem, although the enumeration varies between different textbooks.)

Problem 2: Suppose that R is a ring, that M is a left R-module, and that N is a submodule of M. Let

$$\pi \colon M \to M/N, \ m \mapsto \bar{m}$$

be the canonical projection.

1. If P is a submodule of M with $N \subset P$, show that P/N is a submodule of M/N. (5 points)
2. Show that $(M/N)/(P/N)$ is isomorphic to M/P. (15 points)
3. If U is a submodule of M/N, let $P := \pi^{-1}(U)$ be its preimage. Show that $U = P/N$. (5 points)

(Remark: The second assertion is often called the second isomorphism theorem, while the third assertion is known as the correspondence theorem.)

Problem 3: Show that the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is semisimple if and only if n is squarefree; i.e., if the primes appearing in the factorization of n are all distinct. (25 points)

(Hint: One approach to this problem uses the Jacobson radical. In this and other approaches, the Chinese remainder theorem from elementary number theory can be useful. You may use this form of the Chinese remainder theorem without proof, but you should state it if you use it.)

Problem 4: Suppose that D is a division ring. A matrix

$$A = (a_{ij})_{i,j \leq n} \in M(n \times n, D)$$

is called upper triangular if $a_{ij} = 0$ for $i > j$, and strictly upper triangular if $a_{ij} = 0$ for $i \geq j$.

1. Show that the set of all upper triangular matrices is a subring of the ring $M(n \times n, D)$ of all $n \times n$-matrices with entries in D. (5 points)

2. Show that the Jacobson radical of this subring consists exactly of the strictly upper triangular matrices. (20 points)

Due date: Tuesday, November 14, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.