Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 7**

Ring Theory

Problem 1: Consider the ring

$$R := \left\{ \begin{pmatrix} q & r \\ 0 & n \end{pmatrix} \middle| q, r \in \mathbb{Q}, n \in \mathbb{Z} \right\}$$

R is a subring of the ring $M(2 \times 2, \mathbb{Q})$ of real 2×2-matrices. (You do not need to show that.)

1. Show that R is left noetherian, but not left artinian. (10 points)
2. Show that R is neither right artinian nor right noetherian. (15 points)

(Hint: As in Problem 3 on Sheet 6, note that this ring is a triangular ring in the sense of Problem 2 on Sheet 3, because the analogy with the triangular matrices given there is in this case not only formal. As a consequence, the description of left and right ideals in Problem 3 on Sheet 3 applies.)

Problem 2: Suppose that R is a ring and that M is a left R-module.

1. Show that M^n is a left module over $M(n \times n, R)$ with respect to the operation
$$(a_{ij})_{i,j \leq n}(m_j)_{j \leq n} := \Big(\sum_{j=1}^{n} a_{ij} m_j \Big)_{i \leq n}$$
(You do not need to show that M^n is an abelian group.) (5 points)
2. If M' is a left module over $M(n \times n, R)$, show that $M' \cong M^n$ for some left R-module M. (20 points)

(Hint: One of several possible approaches to the second part uses Problem 1 on Sheet 2 by observing that the diagonal matrices form a subring of $M(n \times n, R)$ which is isomorphic to R^n.)

Problem 3: In the situation of Problem 2, show the following:

1. If N' is a submodule of the $M(n \times n, R)$-module M^n considered in Part 1 of Problem 2, show that there is an R-submodule N of M such that $N' \cong N^n$. (15 points)
2. Show that M is simple as an R-module if and only if M^n is simple as an $M(n \times n, R)$-module. (5 points)
3. Show that the Jacobson radicals are related by the formula
$$J(M(n \times n, R)) = M(n \times n, J(R))$$
(5 points)

Problem 4: Suppose that R is a ring, that L is a nil left ideal, and that I is a nil two-sided ideal. Show that $L + I$ is a nil left ideal. (25 points)

(Hint: Consider the quotient ring R/I.)

Due date: Tuesday, November 7, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.