Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 6**

Ring Theory

Problem 1: Suppose that R is a ring, that M is a left R-module, and that $N \subset M$ is a submodule. Suppose that N and M/N are artinian. Show that M is artinian.

(Remark: This is the converse of Problem 2 on the midterm examination. The same statement is true for noetherian modules instead of artinian modules, but you do not need to show that.) (25 points)

Problem 2: Suppose that the R-module M has a composition series, i.e., a series

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M$$

of submodules with the property that M_i/M_{i-1} is simple for all $i = 1, \ldots, n$. Show that M is artinian.

(Hint: Use Problem 1 and induction on n. This is a partial converse of Problem 3 on Sheet 4 and was already mentioned there. Under these assumptions, M is also noetherian, giving the full converse. However, you do not need to show that.) (25 points)

Problem 3: Consider the ring

$$R := \left\{ \begin{pmatrix} r & s \\ 0 & q \end{pmatrix} \middle| r, s \in \mathbb{R}, q \in \mathbb{Q} \right\}$$

R is a subring of the ring $M(2 \times 2, \mathbb{R})$ of real 2×2-matrices. (You do not need to show that.)

1. Show that $\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 = R$, where

$$M_1 := \left\{ \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \middle| s \in \mathbb{R} \right\} \quad \text{and} \quad M_2 := \left\{ \begin{pmatrix} 0 & s \\ 0 & q \end{pmatrix} \middle| s \in \mathbb{R}, q \in \mathbb{Q} \right\}$$

   is a composition series for R as a left module over itself. (10 points)

2. Show that R is left artinian. (5 points)

3. Show that R is neither right artinian nor right noetherian. (10 points)

(Hint: Note that this ring is a triangular ring in the sense of Problem 2 on Sheet 3, because the analogy with the triangular matrices given there is in this case not only formal. As a consequence, the description of left and right ideals in Problem 3 on Sheet 3 applies.)

Problem 4: Consider the rational numbers $\mathbb{Q}$ as a (left) module over the ring of integers $\mathbb{Z}$.

1. Show that the only $\mathbb{Z}$-submodules of $\mathbb{Q}$ that are invariant under all endomorphisms in $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$ are $\{0\}$ and $\mathbb{Q}$ itself. (15 points)

2. Show that $\mathbb{Q}$ is not semisimple as a $\mathbb{Z}$-module. (10 points)

(Remark: Note that this problem refutes Exercise 11 to Chapter 1 on page 49 of the textbook.)

Due date: Tuesday, October 31, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.