Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 5**

Ring Theory

Problem 1: Suppose that K is a field and that V is a vector space of countably infinite dimension over K. Show that

$$I := \{f \in \mathrm{End}_K(V) \mid \dim f(V) < \infty\}$$

is a two-sided ideal of the ring $R := \mathrm{End}_K(V)$.

(Remark: You need to show in particular that I is an additive subgroup. For background on infinite-dimensional vector spaces, see Chapter IX in N. Jacobson, Lectures in abstract algebra II: Linear algebra, Grad. Texts Math., Vol. 31, Springer, Berlin, 1975.) (25 points)

Problem 2: With the notation of Problem 1, show that the ring R/I is simple.

(Hint: If $f \notin I$, choose a complement to $\ker(f)$, which is then isomorphic to $\mathrm{im}(f)$, which is in turn isomorphic to V. Use these isomorphisms to construct endomorphisms g and h of V with $g \circ f \circ h = \mathrm{id}_V$). (25 points)

Problem 3: Let V be a vector space over the field K with a basis v_{ij}, where $i, j \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For $n \in \mathbb{N}$, define the set

$$L_n := \{f \in \mathrm{End}_K(V) \mid f(v_{ij}) = 0 \text{ for all } i \leq n \text{ and } j \in \mathbb{N}\}$$

1. Show that L_n is a left ideal of the ring $R := \mathrm{End}_K(V)$. (7 points)
2. If
$$I := \{f \in \mathrm{End}_K(V) \mid \dim f(V) < \infty\}$$
and $\pi\colon R \to R/I$ is the canonical projection, show that $\pi(L_n) \supsetneqq \pi(L_{n+1})$. (15 points)
3. Show that R/I is not left artinian. (3 points)

Problem 4: Suppose that p is a prime number. Inside the quotient group $\mathbb{Q}/\mathbb{Z}$, consider the subgroup

$$G_p := \{\bar{x} \in \mathbb{Q}/\mathbb{Z} \mid p^n \bar{x} = \bar{0} \text{ for some } n \in \mathbb{N}\}$$

of elements annihilated by a power of p. The group G_p is called the Prüfer group corresponding to p.

1. Show that, for every $n \in \mathbb{N}$, the element $\overline{1/p^n}$ generates a cyclic subgroup of G_p of order p^n. (3 points)
2. Show that every proper subgroup of G_p is of the above form; in particular, each proper subgroup is finite. (15 points)
3. Show that G_p, considered as a $\mathbb{Z}$-module, is artinian. (2 points)
4. Show that G_p, considered as a $\mathbb{Z}$-module, is not noetherian. (5 points)

(Remark: Note that abelian groups are essentially the same as $\mathbb{Z}$-modules.)

Due date: Tuesday, October 24, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.