Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 4**

Ring Theory

Problem 1: A left module M over the ring R is called artinian if it satisfies the descending chain condition: For a descending chain

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq M_4 \supseteq \dots$$

of submodules of M, there exists a number $i \in \mathbb{N}$ such that $M_j = M_i$ for all $j \geq i$. Show that M is artinian if and only if every nonempty set of submodules of M contains a minimal element. (25 points)

Problem 2: A left module M over the ring R is called noetherian if it satisfies the ascending chain condition: For an ascending chain

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq M_4 \subseteq \dots$$

of submodules of M, there exists a number $i \in \mathbb{N}$ such that $M_j = M_i$ for all $j \geq i$. Show that the following assertions are equivalent:

1. M is noetherian.
2. Every nonempty set of submodules of M contains a maximal element.
3. Every submodule of M is finitely generated. (25 points)

Problem 3: Suppose that M is an R-module that is both artinian and noetherian. Show that M has a composition series, i.e., a series

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_{n-1} \subseteq M_n = M$$

of submodules with the property that M_i/M_{i-1} is simple for all $i = 1, \dots, n$.

(Remark: The converse of this statement is also true.) (25 points)

Problem 4: Consider the ring of integers $\mathbb{Z}$ as a module over itself. Show that this module is noetherian, but not artinian. (25 points)

Due date: Thursday, October 12, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.