Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 3**

Ring Theory

Problem 1: Suppose that M is a left module over the ring R, and that N is a submodule. Show that the following statements are equivalent:

1. N is a direct summand, i.e., there is a second submodule P so that
$$M = N \oplus P$$
is the internal direct sum of the two submodules N and P.
2. The natural inclusion map $i\colon N \to M$ has a left inverse; i.e., there is an R-module homomorphism $j\colon M \to N$ with $j \circ i = \mathrm{id}_N$.
3. The natural projection map $p\colon M \to M/N$ has a right inverse; i.e., there is an R-module homomorphism $q\colon M/N \to M$ with $p \circ q = \mathrm{id}_{M/N}$.

(25 points)

Problem 2: Suppose that R and S are rings. An R-S-bimodule is a left R-module M that is simultaneously a right S-module in such a way that the equation
$$(rm)s = r(ms)$$
holds for all $r \in R$, $s \in S$, and $m \in M$. The external direct sum $T := S \oplus M \oplus R$ is an abelian group with respect to componentwise addition (you do not need to show that). Show that T becomes a ring with respect to the multiplication
$$(s, m, r)(s', m', r') = (ss', rm' + ms', rr')$$
for $r, r' \in R$, $m, m' \in M$, and $s, s' \in S$.

(Remark: T is called the triangular ring associated with M, because the above multiplication looks formally like matrix multiplication if the element (s, m, r) is written formally as
$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$$
a triangular matrix.) (25 points)

Problem 3: Consider the triangular ring T introduced above.

1. If I is a left ideal of T, show that I is the external direct sum $I = I' \oplus I''$, where I' is a left ideal of S and I'' is an R-submodule of $M \oplus R$ with the property that $(ms, 0) \in I''$ for all $m \in M$ and all $s \in I'$. (15 points)
2. If J is a right ideal of T, show that J is the external direct sum $J = J' \oplus J''$, where J'' is a right ideal of R and J' is an S-submodule of $S \oplus M$ with the property that $(0, rm) \in J'$ for all $m \in M$ and all $r \in J''$. (10 points)

(Hint: Consider suitable idempotents.)

Problem 4: Consider the set of complex matrices

$$\mathbb{H} := \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \middle| z, w \in \mathbb{C} \right\}$$

with basis

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad K := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

from Problem 4 on Sheet 2.

1. Show that the map

$$^{-} : \mathbb{H} \to \mathbb{H}, \ A = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mapsto \bar{A} := \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}$$

   is a ring antihomomorphism; i.e., a ring homomorphism from $\mathbb{H}$ to $\mathbb{H}^o$. (15 points)

2. Show that $A\bar{A} = \bar{A}A = \det(A)E$. (5 points)

3. Show that $\mathbb{H}$ is a division algebra, i.e., that every nonzero element has a multiplicative inverse. (5 points)

Due date: Tuesday, October 3, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.