Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 2**

Ring Theory

Problem 1: Suppose that $R_1, R_2, \dots, R_n$ are rings and that M_i is a (left) module over R_i, for $i = 1, \dots, n$. We have seen in class that

$$M := M_1 \times M_2 \times \ldots \times M_n$$

is a (left) module over

$$R := R_1 \times R_2 \times \ldots \times R_n$$

with respect to componentwise addition and multiplication by scalars.

Conversely, suppose that M is a (left) module over $R := R_1 \times R_2 \times \ldots \times R_n$, and set $M_i := e_i M$, where $e_i := (0, \dots, 0, 1, 0, \dots 0)$. Show that M_i is a module over R_i and that

$$M \cong M_1 \times M_2 \times \ldots \times M_n$$

(25 points)

Problem 2: Suppose that R is a ring.

1. The opposite ring R^o is the same set which carries the same additive group structure as R, but the so-called opposite multiplication defined by

   $$r \cdot_o s := s \cdot r$$

   Show that R^o is indeed a ring by verifying the ring axioms. (13 points)

2. Suppose that M is a right R-module. Show that M becomes left R^o-module by defining

   $$r \cdot_o m := m \cdot r$$

   while the additive group structure of M is unchanged. (12 points)

Problem 3: Suppose that R is a ring and that $e \in R$ is an idempotent.

1. Show that

   $$eRe := \{ere \mid r \in R\}$$

   is a ring with unit element e. (5 points)

2. For a left R-module M, show that

   $$\mathrm{Hom}_R(Re, M) \to eM,\ f \mapsto f(e)$$

   is a bijection. (10 points)

3. $\mathrm{Hom}_R(Re, Re)$ is a ring under pointwise addition and composition as multiplication. (You do not need to show that.) Show that this ring is isomorphic to $(eRe)^o$. (10 points)

Problem 4: Consider the set of complex matrices

$$\mathbb{H} := \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \middle| z, w \in \mathbb{C} \right\}$$

1. Show that $\mathbb{H}$ is a subring of the ring of complex 2×2-matrices. Show also that it is closed under multiplication by real numbers, but not under multiplication by complex numbers. (12 points)
2. Show that the elements

   $$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad K := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

   form a basis of $\mathbb{H}$ as a vector space over the real numbers $\mathbb{R}$. Make a multiplication table for these basis elements. (13 points)

Due date: Tuesday, September 26, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.