Memorial University of Newfoundland **Fall Semester 2023**
Yorck Sommerhäuser **MATH 4320/6321: Sheet 1**

Ring Theory

Problem 1: An element e of a ring R is called an idempotent if $e^2 = e$. Two idempotents e and e' in R are called orthogonal if $ee' = e'e = 0$. A family $e_1, e_2, \ldots, e_n$ of pairwise orthogonal idempotents is called complete if

$$e_1 + e_2 + \ldots + e_n = 1$$

1. For a complete family of orthogonal idempotents, show that

   $$R = Re_1 \oplus Re_2 \oplus \ldots \oplus Re_n$$

   is a decomposition into an (internal) direct sum of left ideals. (10 points)

2. Conversely, suppose that

   $$R = L_1 \oplus L_2 \oplus \ldots \oplus L_n$$

   is a decomposition into an (internal) direct sum of left ideals. Show that there is a complete family $e_1, e_2, \ldots, e_n$ of pairwise orthogonal idempotents such that $L_i = Re_i$. (15 points)

Problem 2: An idempotent e in a ring R is called central if it is contained in the centre of the ring, i.e., if it satisfies $er = re$ for all $r \in R$. Consider a complete family $e_1, e_2, \ldots, e_n$ of pairwise orthogonal idempotents. Show that these idempotents are central if and only if the left ideals Re_i are two-sided. (25 points)

Problem 3: Suppose that $R_1, R_2, \ldots, R_n$ are rings. As discussed in class, the cartesian product

$$R = R_1 \times R_2 \times \ldots \times R_n$$

becomes a ring with respect to componentwise addition and multiplication (you do not need to show that). Show that $e_1, e_2, \ldots, e_n$, where

$$e_i := (0, \ldots, 0, 1, 0, \ldots 0)$$

is a complete family of central orthogonal idempotents. (25 points)

Problem 4: Conversely, if $e_1, e_2, \ldots, e_n$ is a complete family of central orthogonal idempotents in a ring R, show that $R_i := Re_i$ is a ring with unit element e_i and that

$$R \cong R_1 \times R_2 \times \ldots \times R_n$$

as rings. (25 points)

Due date: Tuesday, September 19, 2023. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.