Memorial University of Newfoundland **Fall Semester 2020**
Yorck Sommerhäuser **MATH 3370: Sheet 1**

Introductory Number Theory

Problem 1:

1. Prove that
$$x^{2n} - x^{2n-1} + \ldots + x^2 - x + 1 = \frac{x^{2n+1}+1}{x+1}.$$
(10 points)

2. If $2^a + 1$ is prime, prove that $a = 2^n$ for some $n \in N$. (Hint: Part 1 gives a formula for factoring $x^m + 1$ for m odd.) (15 points)

Problem 2: Let $F_n = 2^{2^n} + 1$.

1. Prove using mathematical induction that
$$F_n = 2 + F_0 \cdot F_1 \cdots F_{n-1}$$
(12 points)

2. Prove that, $m \neq n$, there is no natural number $d \neq 1$ that divides both F_m and F_n. (13 points)

(The second part is known as Goldbach's theorem.)

Problem 3: Prove that no perfect number is a power of a prime. (25 points)

Problem 4: Recall that the Fibonacci numbers are defined recursively by the conditions $f_1 = 1 = f_2$ and
$$f_n := f_{n-1} + f_{n-2}$$
if $n \geq 3$. Prove Binet's formula
$$f_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$$
for all $n \geq 1$, where $\alpha := \frac{1+\sqrt{5}}{2}$ and $\beta := \frac{1-\sqrt{5}}{2}$. (25 points)

Due date: Monday, September 21, 2020. Write your solution on letter-sized paper and send your solution back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.