Memorial University of Newfoundland **Winter Semester 2018**
Yorck Sommerhäuser **MATH 6324: Sheet 9**

Lie Algebras

Problem 1: Suppose that K is a field of characteristic zero, that l is a natural number, and that $n := l + 1$. Inside the Lie algebra $L := \mathrm{sl}(n, K)$, consider the subalgebra H of diagonal matrices with its basis $h_i := E_{ii} - E_{i+1,i+1}$ for $i = 1, \ldots, l$.

1. Show that H is a maximal toral subalgebra. (15 points)
2. For $j, k = 1, \ldots, n$ with $j \neq k$, define the linear functionals $\alpha_{jk} \in H^*$ by

   $$\alpha_{jk}(h_i) := \delta_{ik} - \delta_{ij} - \delta_{i+1,k} + \delta_{i+1,j}$$

   Show that these linear functionals are precisely the roots of L. (10 points)

(Remark: Although this appears to be logically inconsistent, it is in a sense better to to Part 2 before Part 1.)

Problem 2: In the situation of Problem 1, assume that $l = 2$, so that $n = 3$.

1. If κ is the Killing form of L, find the 2×2-matrix

   $$\begin{pmatrix} \kappa(h_1, h_1) & \kappa(h_1, h_2) \\ \kappa(h_2, h_1) & \kappa(h_2, h_2) \end{pmatrix}$$

   (8 points)
2. For each of the six pairs (j, k) with $j \neq k$ considered in Problem 1, find the element

   $$t_{jk} := t_{\alpha_{jk}} \in H$$

   (4 points)
3. Find the 6×6-matrix whose entries are

   $$\kappa(t_{jk}, t_{im}) = \alpha_{jk}(t_{im}) = (\alpha_{jk}, \alpha_{im})$$

   (5 points)
4. Using a sheet of paper as a model for H^*, draw a picture of the root system

   $$\Phi := \{\alpha_{jk} \mid j, k = 1, 2, 3; j \neq k\}$$

   in such a way that the length of the roots and the angles between the roots are depicted correctly. (8 points)

Problem 3: For the Lie algebra $L := \mathrm{so}(5, K)$ considered in Problem 2 on Sheet 4, $H := \mathrm{Span}(H_1, H_2)$ is a maximal toral subalgebra. (Although this is not difficult to see, you do not need to show that.) There are eight roots α_{B_1}, α_{B_2}, α_{C_1}, α_{C_2}, α_X, α_Y, α_R, and α_S corresponding to each of the basis elements introduced in that problem.

1. For each of these eight roots α, find the values $\alpha(H_1)$ and $\alpha(H_2)$. (5 points)

2. Show that, out of these eight roots, you can choose a basis of H^* with the property that every other root can be written as an integral linear combination of the basis elements in which either all coefficients are nonnegative or all coefficients are nonpositive. For each root, give the corresponding linear combination explicitly. (10 points)

3. Using a sheet of paper as a model for H^*, draw a picture of the root system so that each of the roots appears as the correct linear combination of the two roots that you have chosen in the first part. (You do not need to match the length and the angles of the roots in this problem, although you might do this automatically.) (10 points)

Problem 4: Suppose that K is an algebraically closed field of characteristic zero, that L is a finite-dimensional semisimple Lie algebra over K, and that H is a maximal toral subalgebra. Show that H is a Cartan subalgebra.

(25 points)

Due date: Thursday, March 29, 2018. Please write your solution on letter-sized paper, and write your name on your solution. It is not necessary to copy down the problems again or to submit this sheet with your solution.