Memorial University of Newfoundland **Winter Semester 2025**
Yorck Sommerhäuser **MATH 4321/6320: Sheet 9**

Group Theory

Problem 1: Determine under which conditions on n the dihedral group D_{2n} of order $2n$ is nilpotent. (Prove all your assertions in full detail.) (25 points)

Problem 2: Suppose that $G_1, \ldots, G_n$ are subgroups of a group G. Generalizing the definitions made on page 25 of the textbook, we define

$$[G_1, \ldots, G_n] := [[\ldots[G_1, \ldots, G_{n-2}], G_{n-1}], G_n]$$

Moreover, for elements $g_1, \ldots, g_n \in G$ we define

$$[g_1, \ldots, g_n] := [[\ldots[g_1, \ldots, g_{n-2}], g_{n-1}], g_n]$$

If $G_1, \ldots, G_n$ are normal, show that

$$[G_1, \ldots, G_n] := \langle [g_1, \ldots, g_n] \mid g_1 \in G_1, \ldots, g_n \in G_n \rangle$$

(Hint: Use induction on n and the commutator formulas in 1.5.4, or rather their variant for our definition of commutators.) (25 points)

Problem 3: If $G = S_5$ is the symmetric group on five letters, consider the subgroups

$$G_1 = \langle (2,3) \rangle \qquad G_2 = \langle (1,2) \rangle \qquad G_3 = \langle (3,4) \rangle \qquad G_4 = \langle (2,5) \rangle$$

of order 2.

1. Show that $[g_1, g_2, g_3, g_4] = 1$ for all $g_1 \in G_1$, $g_2 \in G_2$, $g_3 \in G_3$, and $g_4 \in G_4$. (10 points)
2. Show that $[G_1, G_2, G_3, G_4] \neq \{1\}$. (15 points)

Problem 4: Consider any field K, finite or infinite. For the subgroup

$$\left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in K \right\}$$

of $\mathrm{GL}(3, K)$, find the ascending central series and the descending central series, and decide whether the group is nilpotent. (25 points)
(Remark: This groups appears in many different areas of mathematics and has different names there. In some areas, it is called the Heisenberg group.)

Due date: Thursday, April 3, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.