**Memorial University of Newfoundland** **Winter Semester 2025**
**Yorck Sommerhäuser** **MATH 4321/6320: Sheet 8**

# Group Theory

**Problem 1:** Suppose that $p$ and $q$ are primes and that $p$ divides $q-1$. For two monomorphisms $\varphi : C_p \to \mathrm{Aut}(C_q)$ and $\psi : C_p \to \mathrm{Aut}(C_q)$, show that the semidirect products

$$C_q \rtimes_\varphi C_p \qquad \text{and} \qquad C_q \rtimes_\psi C_p$$

are isomorphic. (25 points)

**Problem 2:** Suppose that $p$ and $q$ are distinct primes. Suppose that $G$ is a nonabelian group of order $pq^2$, where $p > q$. If there is more than one $p$-Sylow subgroup, show that $q = 2$ and $p = 3$, so that $|G| = 12$. (25 points)

**Problem 3:** Suppose that $G$ is a nonabelian group of order $4p$, where $p \geq 5$ is a prime.

1. Suppose that the 2-Sylow subgroups of $G$ are isomorphic to $\mathbb{Z}_4$. Show that $G$ is isomorphic to a semidirect product

$$G \cong \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_4$$

   where $\varphi : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism that is not trivial (i.e., not constantly equal to the unit element). (10 points)

2. Suppose further that the kernel of $\varphi$ has order 2. Show that $G$ contains elements $x$ and $y$ that generate $G$ and satisfy

$$x^{2p} = 1 \qquad x^p = y^2 \qquad xy = yx^{-1}$$

   but not $x^p = 1$. (Remark: This group is called the dicyclic group of order $4p$.) (15 points)

**Problem 4:** Suppose that $p$ is a prime that satisfies $p \equiv 1 \pmod 4$.

1. Show that there are exactly two injective homomorphisms $\varphi$ and $\psi$ from $\mathbb{Z}_4$ to $\mathrm{Aut}(\mathbb{Z}_p)$. (5 points)

2. Show that the corresponding semidirect products

$$\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_4 \qquad \text{and} \qquad \mathbb{Z}_p \rtimes_\psi \mathbb{Z}_4$$

   are isomorphic. (5 points)

3. Show that these two isomorphic semidirect products are not isomorphic to the semidirect product considered in the second part of the preceding problem. (15 points)

   (Hint: The reason is not that we have used a different homomorphism from $\mathbb{Z}_4$ to $\mathrm{Aut}(\mathbb{Z}_p)$ there; as we just saw, different homomorphisms can lead to isomorphic semidirect products.)

Due date: Tuesday, March 25, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.