Memorial University of Newfoundland **Winter Semester 2025**
Yorck Sommerhäuser **MATH 4321/6320: Sheet 7**

Group Theory

Problem 1: Suppose that G is a nonabelian group of order 8.

1. Show that G contains an element x of order 4. (5 points)
2. Show that $\langle x \rangle$ is normal. (5 points)
3. Show that any element $y \notin \langle x \rangle$ satisfies $y^2 \in \langle x \rangle$ and $yxy^{-1} = x^{-1}$. (5 points)
4. Show that $G \cong Q_8$ or $G \cong D_8$. (10 points)

Problem 2: Suppose that N and H are groups and that $\psi : H \to \mathrm{Aut}(N)$ is a group homomorphism. We have seen in class that

$$\varepsilon_N : N \to N \rtimes_\psi H, n \mapsto (n, 1) \qquad \text{and} \qquad \varepsilon_H : H \to N \rtimes_\psi H, h \mapsto (1, h)$$

are group homomorphisms into the associated semidirect product $N \rtimes_\psi H$.

1. Show that $\varepsilon_H(h)\varepsilon_N(n)\varepsilon_H(h)^{-1} = \varepsilon_N(\psi(h)(n))$. (5 points)
2. Suppose that $f_N : N \to G$ and $f_H : H \to G$ are group homomorphisms into another group G that satisfy

   $$f_H(h)f_N(n)f_H(h)^{-1} = f_N(\psi(h)(n))$$

   Show that there is a unique group homomorphism $f : N \rtimes_\psi H \to G$ satisfying $f \circ \varepsilon_N = f_N$ and $f \circ \varepsilon_H = f_H$. (20 points)

(Remark: This is the so-called universal property of the semidirect product.)

Problem 3: Suppose that $n \in \mathbb{N}$ a natural number, and consider the group homomorphism $\psi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$ that maps $\bar{0}$ to the identity and $\bar{1}$ to the inversion mapping, so that we have

$$\psi(\bar{1})(\bar{k}) = -\bar{k}$$

1. Show that the semidirect product $\mathbb{Z}_n \rtimes_\psi \mathbb{Z}_2$ is isomorphic to the dihedral group D_{2n} defined in Problem 1 on Sheet 1. (10 points)
2. If n is odd, show that $D_{4n} \cong D_{2n} \times \mathbb{Z}_2$. (15 points)

Problem 4: Suppose that G is a nonabelian group of order $4p$, where $p \geq 5$ is a prime.

1. Show that there is a unique p-Sylow subgroup. (5 points)
2. Suppose that the 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Show that G is isomorphic to a semidirect product
$$G \cong \mathbb{Z}_p \rtimes_\psi (\mathbb{Z}_2 \times \mathbb{Z}_2)$$
where $\psi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism. (5 points)
3. Show that ψ is not injective and not trivial (i.e., not constantly equal to the unit element). Therefore, its kernel has order 2. (5 points)
4. Find a nontrivial element in the center of $\mathbb{Z}_p \rtimes_\psi (\mathbb{Z}_2 \times \mathbb{Z}_2)$. (5 points)
5. Show that $G \cong D_{4p}$. (5 points)

Due date: Tuesday, March 18, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.