Group Theory

Problem 1: Suppose that p is a positive integer, not necessarily prime.

1. Show that every nonnegative integer n can be written in the form
$$n = \sum_{k=0}^{m} a_k p^k$$
for integers $a_0, \ldots, a_m \in \{0, 1, \ldots, p-1\}$. This representation is called the p-adic expansion of n. The case $p = 10$, which is especially popular, is called the decadic expansion.

2. Show that the p-adic expansion of n is unique in the following sense: If $n = \sum_{k=0}^{m} b_k p^k$ is another expansion for integers $b_0, \ldots, b_m \in \{0, 1, \ldots, p-1\}$, then we have $a_i = b_i$ for all $i = 0, \ldots, m$.

The number
$$s(n) = s_p(n) := \sum_{k=0}^{m} a_k$$
is called the digit sum of n. (15 points)

Problem 2: Suppose that p is a prime. Every nonzero rational number q can be written in the form $q = \frac{p^k m}{p^l n}$, where m and n are not divisible by p. The number
$$o(q) = o_p(q) := k - l$$
is called the order of p in q. (This is the exponent of p after cancellation.)

1. For two nonzero rational numbers q, q', show that $o(qq') = o(q) + o(q')$. (5 points)

2. For an integer n with $1 \leq n < p^e$, show that
$$s(n) + s(p^e - n) - s(p^e) = (p-1)(e - o(n))$$
Decide whether the formula also holds for $n = p^e$. (10 points)

3. For every integer $n \geq 1$, show that
$$o(n!) = \frac{1}{p-1}(n - s(n))$$
(20 points)

4. For every integer n with $1 \leq n \leq p^e$, show that
$$o(\binom{p^e}{n}) = e - o(n)$$
(5 points)

Problem 3: Suppose that p is an odd prime.

1. If $e \geq 2$, show for all $2 \leq n \leq p^{e-2}$ that
$$o(\binom{p^{e-2}}{n}p^n) \geq e$$
Decide whether the inequality also holds for $n = 1$. (10 points)

2. For $e \geq 2$, use the binomial theorem to show that
$$(1+p)^{p^{e-2}} \equiv 1 + p^{e-1} \pmod{p^e}$$
(5 points)

3. For $e \geq 1$, show that $(1+p)^{p^{e-1}} \equiv 1 \pmod{p^e}$. (5 points)

4. For $e \geq 1$, show that $\mathrm{Aut}(C_{p^e}) \cong C_{p^{e-1}} \times C_{p-1}$. (5 points)

Problem 4: Suppose that $p = 2$.

1. If $e \geq 3$, show for all $2 \leq n \leq 2^{e-3}$ that
$$o(\binom{2^{e-3}}{n}2^{2n}) \geq e$$
Decide whether the inequality also holds for $n = 1$. (5 points)

2. For $e \geq 3$, use the binomial theorem to show that
$$5^{2^{e-3}} = (1+2^2)^{2^{e-3}} \equiv 1 + 2^{e-1} \pmod{2^e}$$
(5 points)

3. For $e \geq 2$, show that $5^{2^{e-2}} = (1+2^2)^{2^{e-2}} \equiv 1 \pmod{2^e}$. (5 points)

4. For $e \geq 2$, show that $\mathrm{Aut}(C_{2^e})$ is generated by $x \mapsto x^{-1}$ and $x \mapsto x^5$. (5 points)

5. For $e \geq 3$, show that $\mathrm{Aut}(C_{2^e}) \cong C_2 \times C_{2^{e-2}}$, and discuss whether and in which sense this assertion is correct for $e = 2$. (5 points)

Due date: Tuesday, March 11, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.