Memorial University of Newfoundland **Winter Semester 2025**
Yorck Sommerhäuser **MATH 4321/6320: Sheet 3**

Group Theory

For a group G, the higher commutator groups are defined as $G^1 := G'$, $G^2 := G''$, $G^3 := G'''$, and inductively $G^{n+1} := (G^n)'$. A group is called solvable if some higher commutator group is equal to $\{1\}$, i.e., if there exists $n \in \mathbb{N}$ such that $G^n = \{1\}$ (cf. Thm. 6.1.5, p. 123). G is called metabelian if $G'' = \{1\}$.

Problem 1: Suppose that $G = S_3$, the set of bijective mappings from the set $\{1, 2, 3\}$ to itself.

1. List explicitly the elements of the commutator group G'. (10 points)
2. List explicitly the elements of the second commutator group G'' and also of all higher commutator groups. (10 points)
3. Decide whether G is solvable or metabelian. (5 points)

Problem 2: Suppose that $G = D_{2n}$, the dihedral group considered in Problem 1 on Sheet 1.

1. List explicitly the elements of the commutator group G'. (10 points)
2. List explicitly the elements of the second commutator group G'' and also of all higher commutator groups. (10 points)
3. Decide whether G is solvable or metabelian. (5 points)

Problem 3: For an arbitrary group G and an abelian group A, show that the mapping

$$\operatorname{Hom}(G/G', A) \to \operatorname{Hom}(G, A),\ f \mapsto f \circ \pi$$

where $\pi : G \to G/G'$ is the canonical quotient map, is a bijection.

(The group G/G' is called the commutator factor group, or the commutator quotient group.) (20 points)

Problem 4: Let C_n and C_m be cyclic groups of order n and m, respectively.

1. Show that $C_n \times C_m$ is cyclic if m and n are relatively prime. (15 points)
2. Show that $C_n \times C_m$ is not cyclic if m and n are not relatively prime. (15 points)

Due date: Tuesday, February 4, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.