Memorial University of Newfoundland **Winter Semester 2025**
Yorck Sommerhäuser **MATH 4321/6320: Sheet 1**

Group Theory

Problem 1: For a natural number $n \geq 2$, we consider the complex number $\zeta := e^{2\pi i/n}$, and denote the multiplication by ζ by

$$r : \mathbb{C} \to \mathbb{C},\ z \mapsto \zeta z$$

In addition, we denote complex conjugation by

$$s : \mathbb{C} \to \mathbb{C},\ z \mapsto \bar{z}$$

1. Inside the group $\mathrm{Sym}(\mathbb{C})$ of all bijective mappings from $\mathbb{C}$ to itself with identity element $1 = \mathrm{id}_{\mathbb{C}}$, show that

$$r^n = 1 \qquad s^2 = 1 \qquad rs = sr^{-1}$$

(5 points)

2. Prove that the mappings $1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}$ are all distinct. (10 points)

3. Show that the subgroup $D_{2n} := \langle r, s \rangle$ of $\mathrm{Sym}(\mathbb{C})$ generated by r and s contains $2n$ elements. It is called the dihedral group of order $2n$.(10 points)

4. For $n = 5$, draw a rough sketch of the points $1, \zeta, \zeta^2, \zeta^3, \zeta^4$ in the complex plane. (5 points)

Problem 2: Suppose that G is a finite group that is generated by two distinct elements that both have order 2. Show that there exists a natural number $n \geq 2$ such that $G \cong D_{2n}$. (20 points)

Problem 3: The complex-valued matrices

$$I := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \text{and} \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are invertible and therefore define bijective (linear) mappings from $\mathbb{C}^2$ to itself. This also holds for the matrices $K := IJ$ and $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Show that the matrices $E, I, J, K, -E, -I, -J, -K$ are all distinct. (10 points)

2. Show that the subgroup $Q_8 := \langle I, J \rangle$ of $\mathrm{Sym}(\mathbb{C}^2)$ generated by I and J contains 8 elements. It is called the quaternion group. (10 points)

3. Make a group table for Q_8. (10 points)

Problem 4: The groups Q_8 and D_8 both contain eight elements. Decide whether the groups are isomorphic, and justify your decision by a detailed proof. (20 points)

Due date: Tuesday, January 21, 2025. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.