Memorial University of Newfoundland **Fall Semester 2019**
Yorck Sommerhäuser **MATH 4331/6328: Sheet 8**

Galois Theory

Problem 1: Suppose that $K \subset L$ is a finite field extension, which is given as $L = K(\alpha_1, \dots, \alpha_n)$ for elements $\alpha_1, \dots, \alpha_n \in L$. Suppose that $f_i(x) \in K[x]$ is the minimum polynomial of α_i, and let $f(x) := f_1(x)f_2(x) \cdots f_n(x)$. Suppose that $P \supset L$ is a splitting field of f. Show that $P = K(X)$, where

$$X = \{\sigma(\alpha_i) \mid \sigma \in \mathrm{Gal}(P/K), i = 1, \dots, n\}$$

(This problem should be compared with Exercises 83 and 84 on page 75 in the textbook.) (25 points)

Problem 2: Suppose that $K \subset L$ is a pure field extension, i.e., that $L = K(\alpha)$ for an element $\alpha \in L$ with the property that $\alpha^n \in K$. Let $f(x) \in K[x]$ be the minimum polynomial of α, and suppose that $P \supset L$ is a splitting field of f. Show that P is a radical extension of K. (25 points)

(Hint: Use Problem 1.)

Problem 3: Suppose that $K \subset L \subset P$ is a radical tower, i.e., that $L = K(\alpha)$ for an element $\alpha \in L$ with the property that $\alpha^n \in K$, and that $P = L(\beta)$ for an element $\beta \in P$ with the property that $\beta^m \in L$.

Let $f(x) \in K[x]$ be the minimum polynomial of α, $g(x) \in K[x]$ be the minimum polynomial of β, and suppose that $Q \supset P$ is a splitting field of $f(x)g(x)$. Show that Q is a radical extension of K. (25 points)

(Hint: Use Problem 2. This problem can be generalized to radical towers with more than two terms, as discussed in Exercise 85 on page 75 of the textbook, or also in Lemma 4.17 of Rotman's book 'Advanced modern algebra'.)

Problem 4:

1. Show that the symmetric group S_n is generated by the neighbour transpositions $(1, 2)$, $(2, 3)$, $(3, 4)$, ... , $(n - 1, n)$. (You can assume that S_n is generated by the set of all transpositions (i, j).) (12 points)
2. If n is prime, show that S_n is generated by an n-cycle together with a transposition. (13 points)

Due date: Wednesday, November 20, 2019. Write your solution on letter-sized paper, and write your name on your solution. Prove every assertion that you make in full detail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences.