Memorial University of Newfoundland **Fall Semester 2019**
Yorck Sommerhäuser **MATH 4331/6328: Sheet 6**

Galois Theory

Problem 1: Suppose that K is a field, that $f(x) \in K[x]$ is an irreducible polynomial, and that L is its splitting field. If α and β are two roots of f in L, show that there is $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) = \beta$. (25 points)

(This is Exercise 79 (i) on page 63 in the textbook, which contains a hint for the problem. One states this result by saying that the Galois group acts transitively on the roots of an irreducible polynomial.)

Problem 2: Suppose that K is a field, that $f(x) \in K[x]$ is a polynomial that is relatively prime to its formal derivative $f'(x)$, and that L is a splitting field of f. Suppose that, for any two roots α and β of f in L, there is $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) = \beta$. Show that f is irreducible over K. (25 points)

(This is a minor variant of Exercise 79 (ii) on page 63 in the textbook, which contains a hint for the problem.)

Problem 3: We have seen in Problem 3 on Sheet 5 that the polynomial

$$f(x) = x^4 - 10x^2 + 1 \in \mathbb{Q}[x]$$

is irreducible. Let K be its splitting field.

1. Show that $\mathrm{Gal}(K/\mathbb{Q})$ contains four elements. (10 points)
2. Every group of order 4 is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Determine to which of these two groups $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic. (As always, prove your claims.) (15 points)

(This is a minor variant of Exercise 80 on page 63 in the textbook, which contains a hint for the problem.)

Problem 4: Suppose that K is a field of characteristic $p > 0$, and suppose that the Frobenius homomorphism

$$K \to K, \ \alpha \mapsto \alpha^p$$

is surjective. Show that every non-constant polynomial $f(x) \in K[x]$ is separable. (25 points)

(This is part of Exercise 76 on page 58 in the textbook. It is not completely easy.)

Due date: Monday, November 4, 2019.