Memorial University of Newfoundland **Fall Semester 2019**
Yorck Sommerhäuser **MATH 4331/6328: Sheet 4**

Galois Theory

Problem 1: Suppose that K is a field.

1. Show that every nonzero polynomial $f = f(x) \in K[X]$ can be factored in the form
$$f(x) = ap_1(x)p_2(x)\cdots p_n(x)$$
where a is a constant and $p_1(x), p_2(x), \ldots, p_n(x)$ are not necessarily distinct irreducible monic polynomials. (10 points)

2. If
$$f(x) = bq_1(x)q_2(x)\cdots q_m(x)$$
is a second such factorization, show that $a = b$, $n = m$, and that there is a permutation $\sigma \in S_n$ such that $q_i = p_{\sigma(i)}$. (15 points)

(This problem is a minor variant of Exercise 51 on page 37 in the textbook. For the second part, use Euclid's lemma, which is Corollary 15 on page 26 in the textbook. The problem roughly proves that the polynomial ring over a field is a so-called unique factorization domain. In the more general case, the prime factors are not exactly equal, but rather can differ by a unit.)

Problem 2: In the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients, consider the ideal $I := (x, 2)$ generated by x and the constant polynomial 2 (in the sense of Problem 4 on Sheet 2).

1. Show that I consists exactly of those polynomials whose constant term is even. (10 points)

2. Show that I is not a principal ideal. (15 points)

(This problem is a minor variant of Exercise 34 on page 21 in the textbook. It should also be compared with Exercise 43 on page 31 there.)

Problem 3: Suppose that K is a field and that R is the set of all polynomials in $K[x]$ without linear term, i.e., the set of sequences
$$(c_0, c_1, c_2, c_3, \ldots) \in K[x]$$
with $c_1 = 0$.

1. Show that R is a subring of $K[x]$. (8 points)

2. Explain how a greatest common divisor of x^5 and x^6 would be defined. (3 points)

3. Show that there is no greatest common divisor of x^5 and x^6. (14 points)

(This is a minor variant of Exercise 40 on page 30 in the textbook. For the second part, you need to adapt the definition on page 24 of the textbook; it does not apply literally. If you are in doubt, you can find the definition in most textbooks on abstract algebra. Also, Problem 1 above is helpful for this one.)

Problem 4: Suppose that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

is a polynomial with integer coefficients, and that the fraction $r/s \in \mathbb{Q}$ is a root of this polynomial. Suppose that the fraction is reduced in the sense that $\gcd(r, s) = 1$. Show that r divides a_0 and that s divides a_n. In particular, a rational zero of a monic integral polynomial is an integer. (25 points)

(This problem, which appears as Exercise 63 on page 43 in the textbook, is known as the rational zero test. It is discussed in many precalculus textbooks.)

Due date: Wednesday, October 16, 2019. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to write your student number on your solution, to copy down the problems again, or to submit this sheet with your solution.