**Memorial University of Newfoundland** **Fall Semester 2019**
**Yorck Sommerhäuser** **MATH 4331/6328: Sheet 3**

# Galois Theory

**Problem 1:** Suppose that $R$ is a commutative ring. On the set $R[[x]]$ of all sequences $(c_n)_{n\in\mathbb{N}_0} = (c_0, c_1, c_2, \ldots)$ with entries in $R$, we define an addition

$$(c_n)_{n\in\mathbb{N}_0} + (d_n)_{n\in\mathbb{N}_0} := (c_n + d_n)_{n\in\mathbb{N}_0}$$

and a multiplication

$$(c_n)_{n\in\mathbb{N}_0} \cdot (d_n)_{n\in\mathbb{N}_0} := \Big(\sum_{k=0}^{n} c_k \cdot d_{n-k}\Big)_{n\in\mathbb{N}_0}$$

1. It is relatively obvious that $R[[x]]$ is an abelian group with respect to addition. You do not need to prove this in detail, but you should say what the additive neutral element is and what the inverse element of a given element is. (2 points)
2. Show that the multiplication is commutative, associative, and admits a neutral element. (12 points)
3. Show that the distributive law holds. Therefore $R[[x]]$ is a ring, called the ring of formal power series with coefficients in $R$. (7 points)
4. Let $R[x]$ be the subset of $R[[x]]$ consisting of those sequences $(c_n)_{n\in\mathbb{N}_0}$ for which there is an $m \in \mathbb{N}_0$ such that $c_k = 0$ for all $k > m$. Show that $R[x]$ is a subring of $R[[x]]$, called the ring of polynomials with coefficients in $R$. (4 points)

(This problem should be compared with Example 3 on page 9 in the textbook.)

**Problem 2:** Suppose that $R$ is a commutative ring. For a formal power series $c = (c_n)_{n\in\mathbb{N}_0} \in R[[x]]$, define its formal derivative as

$$c' := ((n+1)c_{n+1})_{n\in\mathbb{N}_0}$$

For two formal power series $c$ and $d$, show that

1. $(c+d)' = c' + d'$ (5 points)
2. $(c \cdot d)' = c' \cdot d + c \cdot d'$ (20 points)

(This problem is a minor generalization of Exercise 8 on page 12 in the textbook.)

**Problem 3:** Suppose that the commutative ring $R$ is a domain, and let $K$ be the field constructed in Problem 1 and Problem 2 on Sheet 2. This field $K$ is known as the fraction field of $R$, or also as its field of fractions or its quotient field. Show the universal property of $K$:

1. The map $R \to K,\ r \mapsto r/1$ is an injective ring homomorphism. (3 points)
2. If $L$ is another field and $f : R \to L$ is an injective ring homomorphism, then there exists a unique ring homomorphism $g : K \to L$ with the property that $g(r/1) = f(r)$ for all $r \in R$. (22 points)

(Similar, but not identical statements can be found in Example 7 on page 18 and in Exercises 23 and 24 on page 19 in the textbook.)

**Problem 4:** Suppose that $R$ is a commutative ring. Show that there is a unique ring homomorphism $f : \mathbb{Z} \to R$. (25 points)

Due date: Monday, October 7, 2019. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to write your student number on your solution, to copy down the problems again, or to submit this sheet with your solution.

**Change of syllabus:** Effective immediately, my office hours will be on Wednesday and Friday from 4:15 pm to 6:15 pm, instead of Monday and Friday from 4:15 pm to 6:15 pm.