Memorial University of Newfoundland **Fall Semester 2019**
Yorck Sommerhäuser **MATH 4331/6328: Sheet 2**

Galois Theory

Problem 1: Suppose that the commutative ring R is a domain, i.e., that the product of any two nonzero elements of R is nonzero.

1. On the set
$$X := \{(a,b) \in R \times R \mid b \neq 0\}$$
define a relation by requiring that
$$(a,b) \sim (c,d) :\Leftrightarrow ad = bc$$
Show that this relation is an equivalence relation. (12 points)

2. Denote the equivalence class $\overline{(a,b)}$ of $(a,b) \in X$ by a/b, and denote the set of all equivalence classes by K. On K, define an addition by
$$(a/b) + (c/d) := (ad + bc)/bd$$
and a multiplication by
$$(a/b)(c/d) := (ac)/(bd)$$
Show that these two operations are well-defined. (13 points)

(This problem should be compared with Theorem 9 on page 15 in the textbook.)

Problem 2: In the situation of the preceding problem, show the following:

1. With respect to addition, K is an abelian group. In other words, you have to prove the associative law, the commutative law, and exhibit neutral and inverse elements. (13 points)

2. With respect to addition and multiplication, K is a field. (12 points)

Problem 3: Suppose that R is a not necessarily commutative ring.

1. If $(I_j)_{j \in J}$ is a family of two-sided ideals in R, show that their intersection
$$I := \bigcap_{j \in J} I_j$$
is again a two-sided ideal. (13 points)

2. If $X \subset R$ is any subset, show that there is two-sided ideal I with the following two properties:

   (a) $X \subset I$.

   (b) If $J \subset R$ is a two-sided ideal with $X \subset J$, then $I \subset J$.

   I is therefore the smallest two-sided ideal in R that contains X. We call I the two-sided ideal generated by X and denote it by (X). (12 points)

(This is a small modification of Problem 30 on page 20 in the textbook.)

Problem 4: In the situation of Problem 3, suppose that R is commutative and that $X = \{a_1, \ldots, a_n\}$ is a finite set. Show that the ideal generated by X is equal to the set

$$(X) = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_1, r_2, \ldots, r_n \in R\}$$

(This is Problem 31 (ii) on page 20 in the textbook.) (25 points)

Due date: Monday, September 30, 2019. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.