Memorial University of Newfoundland **Fall Semester 2019**
Yorck Sommerhäuser **MATH 4331/6328: Sheet 1**

Galois Theory

Problem 1:

1. For the binomial coefficients
$$\binom{n}{k} := \frac{n!}{k!(n-k)!} \in \mathbb{Q}$$
prove Pascal's rule, which states that
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
This is the rule that underlies Pascal's triangle. Note that it shows inductively that $\binom{n}{k} \in \mathbb{Z}$, not only in $\mathbb{Q}$. (10 points)

2. If R is a commutative ring (always with a unit), show the binomial theorem, which states that
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
for all $a, b \in R$ and $n \geq 0$. Explain how the right-hand side really becomes an element of R. (15 points)

(This is Problem 6 on page 12 in the textbook.)

Problem 2: If p is a prime and $0 < k < p$, show that $\binom{p}{k}$ is divisible by p.
(This is Problem 7 on page 12 in the textbook.) (25 points)

Problem 3: Calculating 'naively' with polynomials, suppose that we expand a product of linear factors in the form
$$(x-a_1)(x-a_2)\cdots(x-a_n) = \sum_{k=0}^{n} c_k x^k$$
Show Vieta's formulas, which state that for $k = 1, \ldots, n$, we have
$$c_{n-k} = (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_k}$$
Here, the sum is taken over all k-tuples of indices between 1 and n that satisfy the stated condition.
(This is stated without a detailed proof on page 10 in the textbook. The proof is not completely easy.) (25 points)

Problem 4: Consider a set G together with a map

$$\cdot\colon G \times G \to G,\ \ (a,b) \mapsto a \cdot b$$

that assigns to any two elements $a, b \in G$ their product $a \cdot b$. Suppose that this product map has the following properties:

1. Associativity: For all $a, b, c \in G$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. Existence of a left unit element: There is a distinguished element $e \in G$ with the property that $e \cdot a = a$ for all $a \in G$.
3. Existence of a left inverse element: For a given element $a \in G$, there is an element $b \in G$ such that $b \cdot a = e$.

Show that G is a group, i.e., that the left unit element is also a right unit element and that the left inverse element is also a right inverse element.
(Hint: Start with the inverses.) (25 points)

Due date: Monday, September 23, 2019. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.