Memorial University of Newfoundland **Winter Semester 2022**
Yorck Sommerhäuser **MATH 4310: Sheet 8**

Complex Function Theory

Problem 1:

1. Determine the angle of intersection between the line $y = x - 1$ and the x-axis $y = 0$. (5 points)
2. Sketch the images of these lines under the transformation $f(z) := 1/z$. (10 points)
3. Determine the angle of intersection between the images of these lines. (10 points)

(Hint: This is a minor variant of Exercise 3 for Section 97 in the textbook, where you will find additional hints. The angle of intersection depends on the direction in which the curves are traced out. These directions should be consistent between Part 1 and Part 3.)

Problem 2: Find a harmonic conjugate for the function

$$u(x, y) = x^3 - 3xy^2$$

(Hint: This is a minor variant of Exercise 1 for Section 99 in the textbook, where you will find additional hints.) (25 points)

Problem 3: Suppose that $f\colon A \to B$ is an analytic function that maps the domain A to the domain B, and suppose that $h\colon B \to \mathbb{R}$ has partial derivatives of first and second order that are continuous. We write elements of A in the form $z = x + iy = (x, y)$ and elements of B in the form $w = u + iv = (u, v)$ and write $f(z) = r(z) + is(z) = (r(z), s(z))$.
Denote the composite function by $H(z) = h(f(z))$.

1. Show that
$$H_x(z) = h_u(f(z))r_x(z) + h_v(f(z))s_x(z)$$
and $H_y(z) = h_u(f(z))r_y(z) + h_v(f(z))s_y(z)$. (5 points)
2. Show that
$$\begin{aligned} H_{xx}(z) &= (h_{uu}(f(z))r_x(z) + h_{uv}(f(z))s_x(z))r_x(z) + h_u(f(z))r_{xx}(z) \\ &\quad + (h_{vu}(f(z))r_x(z) + h_{vv}(f(z))s_x(z))s_x(z) + h_v(f(z))s_{xx}(z) \end{aligned}$$

and

$$\begin{aligned}H_{yy}(z) &= (h_{uu}(f(z))r_y(z) + h_{uv}(f(z))s_y(z))r_y(z) + h_u(f(z))r_{yy}(z)\\ &\quad + (h_{vu}(f(z))r_y(z) + h_{vv}(f(z))s_y(z))s_y(z) + h_v(f(z))s_{yy}(z)\end{aligned}$$

(8 points)

3. Show that $H_{xx}(z) + H_{yy}(z) = (h_{uu}(f(z)) + h_{vv}(f(z)))|f'(z)|^2$. (12 points)

(Hint: This is a minor variant of Exercise 8 for Section 99 in the textbook, where you will find additional hints.)

Problem 4: The function

$$H(x,y) = \left(\ln\left(\sqrt{x^2+y^2}\right)\right)^3 - 3\ln\left(\sqrt{x^2+y^2}\right)\left(\arctan\left(\frac{y}{x}\right)\right)^2$$

is harmonic in the right half plane $x > 0$. Explain how this follows from Problem 2 and Problem 3. (25 points)

(Hint: You do not need to show by explicit calculation that this function is harmonic.)

Due date: Monday, March 28, 2022. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.