Memorial University of Newfoundland **Winter Semester 2022**
Yorck Sommerhäuser **MATH 4310: Sheet 7**

Complex Function Theory

Problem 1: Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is a bijective entire function, and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a linear transformation, i.e., that there are complex numbers $a, b \in \mathbb{C}$ with $f(z) = az + b$, where $a \neq 0$.

(Hint: By assumption, there is a unique complex number z_0 with $f(z_0) = 0$. Explain that the order of this zero must be a finite number n. Show that $f(z)/(z - z_0)^n$ is an entire function without zeros and apply Problem 4 on Sheet 6 to $(z - z_0)^n/f(z)$. Then show that $n = 1$.) (30 points)

Problem 2: Suppose that $f\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a continuous bijective function from the extended complex plane to itself. In particular, if $f(z_1) = \infty$ and $f(\infty) = z_2$, we have that

$$\lim_{z\to z_1} f(z) = \infty \qquad \lim_{z\to\infty} f(z) = z_2$$

We assume that $z_1 \neq \infty$, that $z_2 \neq \infty$, and that f is analytic at every point $z_0 \in \mathbb{C} \cup \{\infty\}$. If $z_0 = \infty$, this means by definition that $f(1/z)$ has a removable singularity at 0, and if $z_0 = z_1$, this means that $1/f(z)$ has a removable singularity at z_1. Show that f is a Möbius transformation.

(Hint: First, show that there is a Möbius transformation g with $g(\infty) = z_1$. Then apply Problem 1 to $f \circ g$. Use also the results of Section 70 in the textbook.) (30 points)

Problem 3: The set of complex numbers $z = x + iy$ where $y = 1$ is a line that is parallel to the x-axis. Sketch the image of this line under the mapping $f(z) = \sin(z)$, and explain how your sketch comes about. (25 points)

Problem 4: Describe a Riemann surface for the function

$$f(z) = \sqrt[3]{z} = z^{1/3}$$

(Hint: Use three sheets.) (15 points)

Due date: Monday, March 21, 2022. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.