Memorial University of Newfoundland **Winter Semester 2022**
Yorck Sommerhäuser **MATH 4310: Sheet 6**

Complex Function Theory

Problem 1: If z_1, z_2, z_3, z_4 are distinct complex numbers, their cross ratio is defined as the complex number

$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Suppose that w_1, w_2, w_3, w_4 are also four distinct complex numbers, and suppose that there is a Möbius transformation f with $f(z_i) = w_i$ for $i = 1, 2, 3, 4$. Show that

$$[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$$

(Hint: Use the results of Section 87 in the textbook.) (25 points)

Problem 2: If z_1, z_2, z_3, z_4 are distinct complex numbers, show that

$$[z_1, z_2, z_3, z_4] = [z_2, z_1, z_4, z_3] = [z_3, z_4, z_1, z_2] = [z_4, z_3, z_2, z_1]$$

(25 points)

Problem 3: If z_1, z_2, z_3, z_4 are distinct complex numbers, show that

1. $[z_1, z_3, z_2, z_4] = 1 - [z_1, z_2, z_3, z_4]$ (10 points)
2. $[z_3, z_2, z_1, z_4] = 1/[z_1, z_2, z_3, z_4]$ (10 points)

Problem 4: Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function, and suppose that there are real positive constants M and r as well as a natural number $n \in \mathbb{N}$ such that $|f(z)| \leq M|z|^n$ if $|z| > r$. Show that f is a polynomial of degree at most n.

(Hint: Explain why f can be represented by a power series, and use Cauchy's inequalities (the lemma in Section 49 on page 165 in the textbook) to show that the higher coefficients in this power series must vanish. Note that the case $n = 0$ of this problem is essentially Liouville's theorem.) (30 points)

Due date: Monday, March 14, 2022. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.