Memorial University of Newfoundland
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Winter Semester 2022
MATH 4310: Sheet 5

Complex Function Theory

Problem 1: The set of points $z = x + iy$ that satisfy $x^2 + (y + 1)^2 = 1$ form a circle of radius 1 with centre at $(0, -1)$. Describe the image of this circle under the map $z \mapsto 1/z$ by both an equation and a picture.

(Hint: Explain how you account for the fact that the circle goes through the origin, but the map is not defined there.) (25 points)

Problem 2: Suppose that the Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$, is the identity transformation, i.e., that $f(z) = z$ for all complex numbers z. Show that $b = 0$, $c = 0$, and $a = d$.

(Remark: Note that, conversely, f is the identity transformation if $b = 0$, $c = 0$, and $a = d$.) (25 points)

Problem 3: Recall that any Möbius transformation f can be considered as a map from the extended complex plane $\mathbb{C} \cup \{\infty\}$ to itself. If we assume that $f(\infty) = \infty$, show that f is a linear transformation, i.e., that $f(z) = az + b$ for suitable complex numbers a and b with $a \neq 0$.

(Hint: Use that $\lim_{z\to\infty} f(z) = \infty$. Recall that we discussed in class that, conversely, linear transformations have the property that $f(\infty) = \infty$.) (25 points)

Problem 4: Suppose that the Möbius transformation f leaves the three points 0, 1, and ∞ of the extended complex plane fixed, i.e., satisfies $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$. Show that f is the identity transformation, i.e., that $f(z) = z$ for all complex numbers z.

(Hint: Use Problem 3.) (25 points)

Due date: Monday, March 7, 2022. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to write down your student number on your solution.